Published by Institute of Physics Publishing for SISSA
Received: *July 18, 2007*
Accepted: *August 9, 2007*
Published: *August 22, 2007*

Lagrangians with electric and magnetic charges of $N = 2$ supersymmetric gauge theories

Mathijs de Vroome and Bernard de Wit

Institute for Theoretical Physics and Spinoza Institute, Utrecht University, Utrecht, The Netherlands
E-mail: M.T.deVroome@phys.uu.nl, B.deWit@phys.uu.nl

Abstract: General Lagrangians are constructed for N=2 supersymmetric gauge theories in four space-time dimensions involving gauge groups with (non-abelian) electric and magnetic charges. The charges induce a scalar potential, which, when the charges are regarded as spurionic quantities, is invariant under electric/magnetic duality. The resulting theories are especially relevant for supergravity, but details of the extension to local supersymmetry will be discussed elsewhere. The results include the coupling to hypermultiplets. Without the latter, it is demonstrated how an off-shell representation can be constructed based on vector and tensor supermultiplets.

Keywords: Supersymmetry and Duality, Extended Supersymmetry, Flux compactifications.

 http://jhep.sissa.it/archive/papers/jhep082007064/jhep082007064.pdf

Contents

1. Introduction **1**

2. Vector multiplets, electric/magnetic duality, and non-abelian charges **3**
2.1 Gauge transformations 7
2.2 Electric and magnetic charges 9

3. The gauge group and the embedding tensor **11**

4. Restoring supersymmetry for non-abelian vector multiplets **16**

5. Hypermultiplets **20**

6. Off-shell structure **24**

7. Summary and discussion **29**

1. Introduction

In four space-time dimensions, theories with abelian gauge fields may have more symmetries than are apparent from the Lagrangian (or the corresponding action). The full invariance group may include symmetries of the combined field equations and Bianchi identities that are not realized at the level of the Lagrangian. This group is a subgroup of the electric/magnetic duality group, which, for n vector fields, is equal to $\mathrm{Sp}(2n, \mathbb{R})$. Under a generic electric/magnetic duality the Lagrangian will in general change, but the new Lagrangian will still lead to an equivalent set of field equations and Bianchi identities. Therefore these different Lagrangians, which do not have to share the same symmetry group, belong to the same equivalence class. When the Lagrangian does not change under a duality (possibly after combining with corresponding transformations of the other fields) one is dealing with an invariance of the theory. To appreciate this feature, it is important to note that a Lagrangian does not transform as a function under duality transformations. In fact the gauge fields before and after the transformation are not related by a local field redefinition. This is the underlying reason why the full invariance is not necessarily reflected by an invariance of the Lagrangian that is induced by transformations of the various fields.

When introducing charges for some of the fields, the standard procedure is to introduce minimal couplings and covariant field strengths in the Lagrangian. This implies that the charges are all electric. The gauge group will therefore be contained in the invariance group of the Lagrangian, so that one cannot necessarily gauge any subgroup of the full invariance group. In that case one has two options. Either one uses electric/magnetic duality to obtain another Lagrangian belonging to the same equivalence class that has a more suitable invariance group in which the desired gauge group can be embedded, or, one uses a recently proposed formalism that incorporates both electric and magnetic charges [1]. The latter allows one to start from any particular Lagrangian belonging to a certain equivalence class, provided that this class contains at least one Lagrangian in which all the charges that one intends to switch on are electric.

In this paper we study general gaugings of $N = 2$ supersymmetric gauge theories, based on vector multiplets and hypermultiplets. It is well known that the introduction of charged fields in a supersymmetric field theory tends to break supersymmetry. To preserve supersymmetry the theory has to be extended with a scalar potential and masslike terms. The goal is to derive these terms in the context of the formalism presented in [1]. It is not the first time that this formalism has been used for four-dimensional supersymmetric theories. In [2] it was successfully applied to $N = 4$ supergravity and in [3] to $N = 8$ supergravity. In this approach the cumbersome procedure according to which the ungauged Lagrangian has to be converted to a suitable electric frame, prior to switching on the charges, is avoided. Moreover, the scalar potential and masslike terms that accompany the gaugings are found in a way that is independent of the electric/magnetic duality frame. By introducing both electric *and* magnetic charges the potential will thus fully exhibit the duality invariances. This is of interest, for example, when studying flux compactifications in string theory, because the underlying fluxes are usually subject to integer-valued rotations associated to the non-trivial cycles of the underlying internal manifold.

The framework of [1] incorporates both electric and magnetic charges and their corresponding gauge fields. The charges are encoded in terms of a so-called embedding tensor, which defines the embedding of the gauge group into the full rigid invariance group. This embedding tensor is treated as a spurionic object, so that the electric/magnetic duality structure of the ungauged theory is preserved after charges are turned on. Besides introducing a set of dual magnetic gauge fields, tensor fields are required that transform in the adjoint representation of the rigid invariance group. These extra fields carry additional off-shell degrees of freedom, but the number of physical degrees of freedom remains the same, owing to extra gauge transformations. Prior to [1] it had already been discovered that magnetic charges tend to be accompanied by tensor fields. An early example of this was presented in [4], and subsequently more theories with magnetic charges and tensor fields were constructed, for instance, in [5 – 7]. However, in these references the gauge groups are abelian.

The starting point of this paper is the expression for $N = 2$ supersymmetric Lagrangians of n vector supermultiplets, labeled by indices $\Lambda = 1, \ldots, n$. This Lagrangian is encoded in terms of a holomorphic function $F(X)$, which, for the abelian case, takes the following form,

$$
\begin{aligned}
\mathcal{L}_0 = {} & \mathrm{i}\,\partial_\mu F_\Lambda\,\partial^\mu \bar{X}^\Lambda + \frac{1}{2}\mathrm{i}\,F_{\Lambda\Sigma}\,\bar{\Omega}_i{}^\Lambda \partial\!\!\!/ \Omega^{i\Sigma} + \frac{1}{4}\mathrm{i}\,F_{\Lambda\Sigma}\,F^{-}_{\mu\nu}{}^{\Lambda}\,F^{-\Sigma\,\mu\nu} - \frac{1}{8}\mathrm{i}\,F_{\Lambda\Sigma}\,Y_{ij}{}^{\Lambda}\,Y^{ij\Sigma} \\
& + \frac{1}{8}\mathrm{i}\,F_{\Lambda\Sigma\Gamma}\,Y^{ij\Lambda}\,\bar{\Omega}_i^\Sigma \Omega_j^\Gamma - \frac{1}{16}\mathrm{i}F_{\Lambda\Sigma\Gamma}\,\bar{\Omega}_i{}^\Lambda \gamma^{\mu\nu}\Omega_j^\Gamma\,\varepsilon^{ij}\,F^{-}_{\mu\nu}{}^{\Sigma} \\
& - \frac{1}{48}\mathrm{i}\,\varepsilon^{ij}\varepsilon^{kl}\,F_{\Lambda\Sigma\Gamma\Xi}\,\bar{\Omega}_i^\Lambda \Omega_k^\Sigma\,\bar{\Omega}_j^\Gamma \Omega_l^\Xi + \text{h.c.}\,,
\end{aligned}
\tag{1.1}
$$

where $F_{\Lambda_1\cdots\Lambda_k}$ denotes the k-th derivative of $F(X)$. The fermion fields Ω^Λ and the auxiliary fields Y^Λ carry SU(2) indices $i,j,\ldots=1,2$. Spinors $\Omega_i{}^\Lambda$ have positive, and spinors $\Omega^{i\Lambda}$ have negative chirality (so that $\gamma^5\Omega_i{}^\Lambda=\Omega_i{}^\Lambda$ and $\gamma^5\Omega^{i\Lambda}=-\Omega^{i\Lambda}$). The auxiliary fields satisfy the pseudo-reality constraint $(Y_{ij}{}^\Lambda)^*=\varepsilon^{ik}\varepsilon^{jl}Y_{kl}^\Lambda$. The tensors $F_{\mu\nu}^{\pm\,\Lambda}$ are the (anti-)selfdual components of the field strengths, which will be expressed in terms of vector fields $A_\mu{}^\Lambda$. Even when all charges are electric it is possible that the function $F(X)$ is not invariant under the gauge group. In that case the gauge group must be non-semisimple [8]. The gauge group for the hypermultiplets can be either abelian or non-abelian, but a non-trivial gauge group for the vector multiplets is always non-abelian, possibly with a central extension.

The supersymmetric Lagrangians derived in this paper incorporate gaugings in both the vector and hypermultiplet sectors. Although the vector multiplets are originally defined as off-shell multiplets, the presence of the magnetic charges causes a breakdown of off-shell supersymmetry. Of course, hypermultiplets are not based on an off-shell representation of the supersymmetry algebra irrespective of the presence of charges. It is an interesting question whether the results of this section can be reformulated such that the vector multiplets retain their off-shell form and, indeed, we show that such an off-shell version can be constructed based on vector and tensor supermultiplets. However, we refrain from considering the extension of the theories of this paper to supergravity. This extension is expected to be straightforward upon use of the superconformal multiplet calculus [8 − 10]. We intend to return to this topic elsewhere.

This paper is organized as follows. In section 2 we recall the relevant features of $N=2$ vector multiplets and electric/magnetic duality, and discuss the introduction of electric and magnetic charges. In section 3 we introduce the embedding tensor and we review the formalism of [1]. Section 4 deals with the restoration of supersymmetry in vector multiplet models after a gauging, and section 5 gives the extension with hypermultiplets. The off-shell formulation of the theories of this paper is discussed in section 6, and in section 7 we summarize the results obtained and briefly indicate some of their applications.

2. Vector multiplets, electric/magnetic duality, and non-abelian charges

In this section we discuss electric/magnetic duality and the introduction of charges for systems of vector supermultiplets. To facilitate the presentation it is convenient to decompose the Lagrangian (1.1) as follows,

$$\mathcal{L}_0=\mathcal{L}_{\text{matter}}+\mathcal{L}_{\text{kin}}+\mathcal{L}_{\Omega^4}+\mathcal{L}_Y\,, \tag{2.1}$$

where $\mathcal{L}_{\text{matter}}$ contains the kinetic terms of the scalar and spinor fields,

$$\begin{aligned}\mathcal{L}_{\text{matter}}=&\ \mathrm{i}\Big(\partial_\mu F_\Lambda\,\partial^\mu\bar X^\Lambda-\partial_\mu\bar F_\Lambda\,\partial^\mu X^\Lambda\Big)\\ &-\frac{1}{4}N_{\Lambda\Sigma}\Big(\bar\Omega^{i\Lambda}\partial\!\!\!/\Omega_i{}^\Sigma+\bar\Omega_i{}^\Lambda\partial\!\!\!/\Omega^{i\Sigma}\Big)-\frac{1}{4}\mathrm{i}\Big(\bar\Omega_i{}^\Lambda\partial\!\!\!/F_{\Lambda\Sigma}\Omega^{i\Sigma}-\bar\Omega^{i\Lambda}\partial\!\!\!/\bar F_{\Lambda\Sigma}\Omega_i{}^\Sigma\Big)\,.\end{aligned}\tag{2.2}$$

The kinetic terms of the vector fields combined with a number of terms that are related to them by electric/magnetic duality, are contained in $\mathcal{L}_{\rm vector}$,

$$
\begin{aligned}
\mathcal{L}_{\rm vector} =\ & \frac{1}{4}\mathrm{i}\Big(F_{\Lambda\Sigma}F^{-\Lambda}_{\mu\nu}F^{-\Sigma\,\mu\nu} - \bar{F}_{\Lambda\Sigma}F^{+\Lambda}_{\mu\nu}F^{+\Sigma\,\mu\nu}\Big)\\
& -\frac{1}{16}\mathrm{i}\Big(F_{\Lambda\Sigma\Gamma}\bar{\Omega}_i^{\Lambda}\,\gamma^{\mu\nu}F^{-\Sigma}_{\mu\nu}\,\Omega_j^{\Gamma}\,\varepsilon^{ij} - \bar{F}_{\Lambda\Sigma\Gamma}\bar{\Omega}^{i\Lambda}\,\gamma^{\mu\nu}F^{+\Sigma}_{\mu\nu}\,\Omega^{j\Gamma}\,\varepsilon_{ij}\Big)\\
& -\frac{1}{256}\mathrm{i}N^{\Delta\Omega}\Big(F_{\Delta\Lambda\Sigma}\bar{\Omega}_i{}^{\Lambda}\gamma_{\mu\nu}\Omega_j{}^{\Sigma}\varepsilon^{ij}\Big)\Big(F_{\Gamma\Xi\Omega}\bar{\Omega}_k{}^{\Gamma}\gamma^{\mu\nu}\Omega_l{}^{\Xi}\varepsilon^{kl}\Big)\\
& +\frac{1}{256}\mathrm{i}N^{\Delta\Omega}\Big(\bar{F}_{\Delta\Lambda\Sigma}\bar{\Omega}^{i\Lambda}\gamma_{\mu\nu}\Omega^{j\Sigma}\varepsilon_{ij}\Big)\Big(\bar{F}_{\Gamma\Xi\Omega}\bar{\Omega}^{k\Gamma}\gamma^{\mu\nu}\Omega^{l\Xi}\varepsilon_{kl}\Big)\,. \qquad (2.3)
\end{aligned}
$$

Quartic spinor terms that are consistent with respect to electric/magnetic duality, are given by

$$
\begin{aligned}
\mathcal{L}_{\Omega^4} =\ & \frac{1}{384}\mathrm{i}\Big(F_{\Lambda\Sigma\Gamma\Xi} + 3\,\mathrm{i}N^{\Delta\Omega}\,F_{\Delta(\Lambda\Gamma}F_{\Sigma\Xi)\Omega}\Big)\,\bar{\Omega}_i{}^{\Lambda}\gamma_{\mu\nu}\Omega_j{}^{\Sigma}\varepsilon^{ij}\,\bar{\Omega}_k{}^{\Gamma}\gamma^{\mu\nu}\Omega_l{}^{\Xi}\varepsilon^{kl}\\
& -\frac{1}{384}\mathrm{i}\Big(\bar{F}_{\Lambda\Sigma\Gamma\Xi} - 3\,\mathrm{i}N^{\Delta\Omega}\bar{F}_{\Delta(\Lambda\Gamma}\bar{F}_{\Sigma\Xi)\Omega}\Big)\,\bar{\Omega}^{i\Lambda}\gamma_{\mu\nu}\Omega^{j\Sigma}\varepsilon_{ij}\,\bar{\Omega}^{k\Gamma}\gamma^{\mu\nu}\Omega^{l\Xi}\varepsilon_{kl}\\
& -\frac{1}{16}N^{\Delta\Omega}F_{\Delta\Lambda\Sigma}\bar{F}_{\Gamma\Xi\Omega}\,\bar{\Omega}^{i\Gamma}\Omega^{j\Xi}\,\bar{\Omega}_i{}^{\Lambda}\Omega_j{}^{\Sigma}\,, \qquad (2.4)
\end{aligned}
$$

and, finally, $\mathcal{L}_Y$ comprises the terms associated with the auxiliary fields $Y_{ij}{}^{\Lambda}$,

$$
\begin{aligned}
\mathcal{L}_Y =\ & \frac{1}{8}N^{\Lambda\Sigma}\,\Big(N_{\Lambda\Gamma}Y_{ij}{}^{\Gamma} + \frac{1}{2}\mathrm{i}(F_{\Lambda\Gamma\Omega}\,\bar{\Omega}_i{}^{\Gamma}\Omega_j{}^{\Omega} - \bar{F}_{\Lambda\Gamma\Omega}\,\bar{\Omega}^{k\Gamma}\Omega^{l\Omega}\varepsilon_{ik}\varepsilon_{jl})\Big)\\
& \times\Big(N_{\Sigma\Xi}Y^{ij\Xi} + \frac{1}{2}\mathrm{i}(F_{\Sigma\Xi\Delta}\,\bar{\Omega}_m{}^{\Xi}\Omega_n{}^{\Delta}\varepsilon^{im}\varepsilon^{jn} - \bar{F}_{\Sigma\Xi\Delta}\,\bar{\Omega}^{i\Xi}\Omega^{j\Delta})\Big)\,. \qquad (2.5)
\end{aligned}
$$

This last result for $\mathcal{L}_Y$ is not obviously consistent with electric/magnetic duality. We return to this in a sequal. Here and henceforth we use the notation,

$$
N_{\Lambda\Sigma} = -\mathrm{i}F_{\Lambda\Sigma} + \mathrm{i}\bar{F}_{\Lambda\Sigma}\,, \qquad N^{\Lambda\Sigma} \equiv [N^{-1}]^{\Lambda\Sigma}\,. \qquad (2.6)
$$

Note that $N_{\Lambda\Sigma}$ plays the role of the inverse effective coupling constants while the real part of $F_{\Lambda\Sigma}$ plays the role of the generalized theta angles.

The non-linear sigma model contained in (2.3) exhibits an interesting geometry known as *special geometry*. The complex scalars X^{Λ} parametrize an n-dimensional target space with metric $g_{\Lambda\bar{\Sigma}} = N_{\Lambda\Sigma}$. This is a Kähler space: its metric equals $g_{\Lambda\bar{\Sigma}} = \partial^2 K(X,\bar{X})/\partial X^{\Lambda}\,\partial\bar{X}^{\Sigma}$, with Kähler potential

$$
K(X,\bar{X}) = \mathrm{i}X^{\Lambda}\,\bar{F}_{\Lambda}(\bar{X}) - \mathrm{i}\bar{X}^{\Lambda}\,F_{\Lambda}(X)\,. \qquad (2.7)
$$

The supersymmetry transformations that leave the action corresponding to (2.1) invariant, are given by

$$
\begin{aligned}
\delta X^{\Lambda} &= \bar{\epsilon}^i\Omega_i{}^{\Lambda}\,,\\
\delta A_{\mu}{}^{\Lambda} &= \varepsilon^{ij}\bar{\epsilon}_i\gamma_{\mu}\Omega_j{}^{\Lambda} + \varepsilon_{ij}\bar{\epsilon}^i\gamma_{\mu}\Omega^{j\,\Lambda}\,,\\
\delta\Omega_i{}^{\Lambda} &= 2\partial\!\!\!/ X^{\Lambda}\epsilon_i + \frac{1}{2}\gamma^{\mu\nu}F^{-}_{\mu\nu}{}^{\Lambda}\varepsilon_{ij}\epsilon^j + Y_{ij}{}^{\Lambda}\epsilon^j\,,\\
\delta Y_{ij}{}^{\Lambda} &= 2\bar{\epsilon}_{(i}\partial\!\!\!/\Omega_{j)}{}^{\Lambda} + 2\varepsilon_{ik}\varepsilon_{jl}\,\bar{\epsilon}^{(k}\partial\!\!\!/\Omega^{l)\Lambda}\,. \qquad (2.8)
\end{aligned}
$$

In the absence of charged fields, abelian gauge fields $A_\mu{}^\Lambda$ appear exclusively through the field strengths, $F_{\mu\nu}{}^\Lambda = 2\,\partial_{[\mu}A_{\nu]}{}^\Lambda$ (we consider Lagrangians that are at most quadratic in derivatives). The field equations for these fields and the Bianchi identities for the field strengths comprise $2n$ equations,

$$\partial_{[\mu}F_{\nu\rho]}{}^\Lambda = 0 = \partial_{[\mu}G_{\nu\rho]\,\Lambda}\,, \tag{2.9}$$

where

$$G_{\mu\nu\,\Lambda} = \varepsilon_{\mu\nu\rho\sigma}\,\frac{\partial\mathcal{L}}{\partial F_{\rho\sigma}{}^\Lambda}\,. \tag{2.10}$$

In the case at hand this implies,

$$G^-_{\mu\nu\Lambda} = F_{\Lambda\Sigma}\,F^{-\,\Sigma}_{\mu\nu} - \frac{1}{8}F_{\Lambda\Sigma\Gamma}\,\bar\Omega_i{}^\Sigma\gamma_{\mu\nu}\Omega_j{}^\Gamma\,\varepsilon^{ij}\,. \tag{2.11}$$

It is convenient to combine the tensors $F_{\mu\nu}{}^\Lambda$ and $G_{\mu\nu\Lambda}$ into a $2n$-dimensional vector,

$$G_{\mu\nu}{}^M = \begin{pmatrix} F_{\mu\nu}{}^\Lambda \\ G_{\mu\nu\Lambda}\end{pmatrix}\,, \tag{2.12}$$

so that (2.9) reads $\partial_{[\mu}G_{\nu\rho]}{}^M = 0$. Obviously these $2n$ equations are invariant under real $2n$-dimensional rotations of the tensors $G_{\mu\nu}{}^M$,

$$\begin{pmatrix} F^\Lambda \\ G_\Lambda\end{pmatrix} \longrightarrow \begin{pmatrix} U^\Lambda{}_\Sigma & Z^{\Lambda\Sigma} \\ W_{\Lambda\Sigma} & V_\Lambda{}^\Sigma\end{pmatrix}\begin{pmatrix} F^\Sigma \\ G_\Sigma\end{pmatrix}\,. \tag{2.13}$$

Half of the rotated tensors can be adopted as new field strengths defined in terms of new gauge fields, and the Bianchi identities on the remaining tensors can then be interpreted as field equations belonging to some new Lagrangian expressed in terms of the new field strengths. In order that such a Lagrangian exists, the real matrix in (2.13) must belong to the group $\mathrm{Sp}(2n;\mathbb{R})$. This group consists of real matrices that leave the skew-symmetric tensor Ω_{MN} invariant,

$$\Omega = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0\end{pmatrix}\,. \tag{2.14}$$

The conjugate matrix Ω^{MN} is defined by $\Omega^{MN}\Omega_{NP} = -\delta^M{}_P$. Here we employ an $\mathrm{Sp}(2n,\mathbb{R})$ covariant notation for the $2n$-dimensional symplectic indices $M,N,\ldots$, such that $Z^M = (Z^\Lambda, Z_\Lambda)$. Likewise we use vectors with lower indices according to $Y_M = (Y_\Lambda, Y^\Lambda)$, transforming according to the conjugate representation so that $Z^M\,Y_M$ is invariant.

The $\mathrm{Sp}(2n;\mathbb{R})$ transformations are known as electric/magnetic dualities, which also act on electric and magnetic charges (for a review of electric/magnetic duality, see [11]). The Lagrangian depends on the electric/magnetic duality frame and is therefore not unique.[1] Different Lagrangians related by electric/magnetic duality lead to equivalent field equations and thus belong to the same equivalence class. These alternative Lagrangians remain

[1] Up to terms proportional to the field equations of the vector fields and the auxiliary fields, the Lagrangian is covariant under electric/magnetic duality.

JHEP08(2007)064

supersymmetric and when applying suitable redefinitions to the other fields, they can again be brought into the form (2.3), characterized by a new holomorphic function $F(X)$. In other words, different functions $F(X)$ can belong to the same equivalence class. The new function is such that the vector $X^M = (X^\Lambda, F_\Lambda)$ transforms under electric/magnetic duality according to

$$\begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{X}^\Lambda \\ \tilde{F}_\Lambda \end{pmatrix} = \begin{pmatrix} U^\Lambda{}_\Sigma & Z^{\Lambda\Sigma} \\ W_{\Lambda\Sigma} & V_\Lambda{}^\Sigma \end{pmatrix} \begin{pmatrix} X^\Sigma \\ F_\Sigma \end{pmatrix} . \tag{2.15}$$

The new function $\tilde{F}(\tilde{X})$ of the new scalars $\tilde{X}^\Lambda$ follows from integration of (2.15) and takes the form

$$\begin{aligned} \tilde{F}(\tilde{X}) = {} & F(X) - \frac{1}{2} X^\Lambda F_\Lambda(X) + \frac{1}{2} (U^{\mathrm{T}} W)_{\Lambda\Sigma} X^\Lambda X^\Sigma \\ & + \frac{1}{2} (U^{\mathrm{T}} V + W^{\mathrm{T}} Z)_\Lambda{}^\Sigma X^\Lambda F_\Sigma(X) + \frac{1}{2} (Z^{\mathrm{T}} V)^{\Lambda\Sigma} F_\Lambda(X) F_\Sigma(X) , \end{aligned} \tag{2.16}$$

up to a constant and to terms linear in the $\tilde{X}^\Lambda$. These terms, which will be ignored in what follows, cannot be present in the case of local supersymmetry. In general it is not easy to determine $\tilde{F}(\tilde{X})$ from (2.16) as it involves the inversion of $\tilde{X}^\Lambda = U^\Lambda{}_\Sigma X^\Sigma + Z^{\Lambda\Sigma} F_\Sigma(X)$. The duality transformations on higher derivatives of $F(X)$ follow by differentiation and we note the results [12],

$$\begin{aligned} \tilde{F}_{\Lambda\Sigma}(\tilde{X}) &= (V_\Lambda{}^\Gamma F_{\Gamma\Xi} + W_{\Lambda\Xi}) \, [\mathcal{S}^{-1}]^\Xi{}_\Sigma , \\ \tilde{F}_{\Lambda\Sigma\Gamma}(\tilde{X}) &= F_{\Xi\Delta\Omega} \, [\mathcal{S}^{-1}]^\Xi{}_\Lambda \, [\mathcal{S}^{-1}]^\Delta{}_\Sigma \, [\mathcal{S}^{-1}]^\Omega{}_\Gamma , \end{aligned} \tag{2.17}$$

where $\mathcal{S}^\Lambda{}_\Sigma = \partial \tilde{X}^\Lambda / \partial X^\Sigma = U^\Lambda{}_\Sigma + Z^{\Lambda\Gamma} F_{\Gamma\Sigma}$. From the first equation one derives,

$$\tilde{N}_{\Lambda\Sigma}(\tilde{X}, \bar{\tilde{X}}) = N_{\Gamma\Delta} \, [\mathcal{S}^{-1}]^\Gamma{}_\Lambda \, [\bar{\mathcal{S}}^{-1}]^\Delta{}_\Sigma . \tag{2.18}$$

To determine the action of the dualities on the fermions, we consider supersymmetry transformations of $X^M = (X^\Lambda, F_\Lambda)$, which take the form $\delta X^M = \bar{\epsilon}^i \Omega_i{}^M$, thus defining an $\mathrm{Sp}(2n, \mathbb{R})$ covariant fermionic vector $\Omega_i{}^M$,

$$\Omega_i{}^M = \begin{pmatrix} \Omega_i{}^\Lambda \\ F_{\Lambda\Sigma} \, \Omega_i{}^\Sigma \end{pmatrix} . \tag{2.19}$$

Complex conjugation leads to a second vector, Ω^{iM}, of opposite chirality. From (2.19) one derives directly that, under electric/magnetic duality,

$$\tilde{\Omega}_i{}^\Lambda = \mathcal{S}^\Lambda{}_\Sigma \, \Omega_i{}^\Sigma . \tag{2.20}$$

With this result one can show that (2.13), (2.17) and (2.20) are consistent.

The supersymmetry transformation of $\Omega_i{}^M$ takes the following form,

$$\delta \Omega_i{}^M = 2 \partial\!\!\!/ X^M \epsilon_i + \frac{1}{2} \gamma^{\mu\nu} G_{\mu\nu}^{-}{}^M \varepsilon_{ij} \epsilon^j + Z_{ij}{}^M \epsilon^j , \tag{2.21}$$

where

$$Z_{ij}{}^{M} = \begin{pmatrix} Y_{ij}{}^{\Lambda} \\ F_{\Lambda\Sigma}\, Y_{ij}{}^{\Sigma} - \tfrac{1}{2} F_{\Lambda\Sigma\Gamma}\, \bar{\Omega}_i{}^{\Sigma} \Omega_j{}^{\Gamma} \end{pmatrix} . \tag{2.22}$$

This suggests that $Z_{ij}{}^{M}$ transforms under electric/magnetic duality as a symplectic vector. However, this is only possible provided we drop the pseudo-reality constraint on $Y_{ij}{}^{\Lambda}$. In that case imposing a pseudo-reality condition on $Z_{ij}{}^{M}$ is manifestly consistent with $\mathrm{Sp}(2n;\mathbb{R})$ and implies both the pseudo-reality of and the field equations associated with the $Y_{ij}{}^{\Lambda}$.

The electric/magnetic duality transformations thus define equivalence classes of Lagrangians. A subgroup thereof may constitute an invariance of the theory [13], meaning that the Lagrangian and its underlying function $F(X)$ do not change [10, 14]. More specifically, an invariance implies

$$\tilde{F}(\tilde{X}) = F(\tilde{X}) \,, \tag{2.23}$$

so that the result of the duality leads to a Lagrangian based on $\tilde{F}(\tilde{X})$ which is identical to the original Lagrangian. Because $\tilde{F}(\tilde{X}) \neq F(X)$, as is obvious from (2.16), $F(X)$ is not an invariant *function*. Instead the above equation implies that the substitution $X^{\Lambda} \to \tilde{X}^{\Lambda}$ into the function $F(X)$ and its derivatives, induces precisely the duality transformations. For example, we obtain,

$$\begin{aligned} F_{\Lambda}(\tilde{X}) &= V_{\Lambda}{}^{\Sigma} F_{\Sigma}(X) + W_{\Lambda\Sigma} X^{\Sigma} \,, \\ F_{\Lambda\Sigma}(\tilde{X}) &= (V_{\Lambda}{}^{\Gamma} F_{\Gamma\Xi} + W_{\Lambda\Xi})\, [\mathcal{S}^{-1}]^{\Xi}{}_{\Sigma} \,, \\ F_{\Lambda\Sigma\Gamma}(\tilde{X}) &= F_{\Xi\Delta\Omega}\, [\mathcal{S}^{-1}]^{\Xi}{}_{\Lambda}\, [\mathcal{S}^{-1}]^{\Delta}{}_{\Sigma}\, [\mathcal{S}^{-1}]^{\Omega}{}_{\Gamma} \,. \end{aligned} \tag{2.24}$$

We elucidate these invariances for the subgroup that acts linearly on the gauge fields $A_\mu{}^{\Lambda}$. These symmetries are characterized by the fact that the matrix in (2.13) and (2.15) has a block-triangular form with $V = [U^{\mathrm{T}}]^{-1}$ and $Z = 0$. Hence this is not a general duality as the Lagrangian is still based on the same gauge fields, up to the linear transformation $A_\mu{}^{\Lambda} \to \tilde{A}_\mu{}^{\Lambda} = U^{\Lambda}{}_{\Sigma} A_\mu{}^{\Sigma}$. Note that all fields in the Lagrangian (2.3) carry upper indices and are thus subject to the same linear transformation. The function $F(X)$ changes with an additive term which is a quadratic polynomial with real coefficients.

$$\tilde{F}(\tilde{X}) = F(U^{\Lambda}{}_{\Sigma} X^{\Sigma}) = F(X) + \frac{1}{2} (U^{\mathrm{T}} W)_{\Lambda\Sigma}\, X^{\Lambda} X^{\Sigma} \,. \tag{2.25}$$

This term induces a total derivative term in the Lagrangian, equal to

$$\mathcal{L} \to \mathcal{L} - \frac{1}{8} \mathrm{i} \varepsilon^{\mu\nu\rho\sigma} (U^{\mathrm{T}} W)_{\Lambda\Sigma}\, F_{\mu\nu}{}^{\Lambda} F_{\rho\sigma}{}^{\Sigma} \,. \tag{2.26}$$

2.1 Gauge transformations

Non-abelian gauge groups will act non-trivially on the vector fields and must therefore involve a subgroup of the duality group. The electric gauge fields $A_\mu{}^{\Lambda}$ associated with this gauge group are provided by vector multiplets. Because the duality group acts on both electric and magnetic charges, in view of the fact that it mixes field strengths with dual field strengths as shown by (2.13), we will eventually introduce magnetic gauge fields

$A_{\mu\Lambda}$ as well, following the procedure explained in [1]. The $2n$ gauge fields $A_\mu{}^M$ will then comprise both type of fields, $A_\mu{}^M = (A_\mu{}^\Lambda, A_{\mu\Lambda})$. The role played by the magnetic gauge fields will be clarified later. For the moment one may associate $A_{\mu\,\Lambda}$ with the dual field strengths $G_{\mu\nu\,\Lambda}$, by writing $G_{\mu\nu\,\Lambda} \equiv 2\,\partial_{[\mu}A_{\nu]\Lambda}$.

The generators (as far as their embedding in the duality group is concerned) are defined as follows. The generators of the subgroup that is gauged, are $2n$-by-$2n$ matrices T_M, where we are assuming the presence of both electric and magnetic gauge fields, so that the generators decompose according to $T_M = (T_\Lambda, T^\Lambda)$. Obviously $T_{\Lambda N}{}^P$ and $T^\Lambda{}_N{}^P$ can be decomposed into the generators of the duality group and are thus of the form specified in (2.13). Denoting the gauge group parameters by $\Lambda^M(x) = (\Lambda^\Lambda(x), \Lambda_\Lambda(x))$, $2n$-dimensional $\mathrm{Sp}(2n;\mathbb{R})$ vectors Y^M and Z_M transform according to

$$\delta Y^M = -g\Lambda^N\,T_{NP}{}^M\,Y^P\,, \qquad \delta Z_M = g\Lambda^N\,T_{NM}{}^P\,Z_P\,, \tag{2.27}$$

where g denotes a universal gauge coupling constant. Covariant derivatives thus take the form,

$$\begin{aligned} D_\mu Y^M &= \partial_\mu Y^M + gA_\mu{}^N\,T_{NP}{}^M\,Y^P \\ &= \partial_\mu Y^M + gA_\mu{}^\Lambda\,T_{\Lambda P}{}^M\,Y^P + gA_{\mu\Lambda}\,T^\Lambda{}_P{}^M\,Y^P\,, \end{aligned} \tag{2.28}$$

and similarly for $D_\mu Z_M$. The gauge fields then transform according to

$$\delta A_\mu{}^M = \partial_\mu\Lambda^M + g\,T_{PQ}{}^M A_\mu{}^P\,\Lambda^Q\,. \tag{2.29}$$

For clarity we first consider electric gaugings where the gauge transformations have a block-triangular form and there are only electric gauge fields. Hence we ignore the fields $A_{\mu\Lambda}$ and assume $T^\Lambda{}_N{}^P = 0$ and $T_\Lambda{}^{\Sigma\Gamma} = 0$. All the fields in the Lagrangian carry upper indices, so that they will transform as in $\delta X^\Lambda = -g\Lambda^\Gamma T_{\Gamma\Sigma}{}^\Lambda\,X^\Sigma$. The transformation rule for $A_\mu{}^\Lambda$ given above is in accord with this expression, provided we assume that $T_{\Gamma\Sigma}{}^\Lambda$ is antisymmetric in Γ and Σ. This has to be the case here as consistency requires that the $T_{\Gamma\Sigma}{}^\Lambda$ are structure constants of the non-abelian group. In the more general situation discussed in later sections, this is not necessarily the case. The embedding into $\mathrm{Sp}(2n,\mathbb{R})$ implies furthermore that $T_{\Lambda\Sigma}{}^\Gamma = -T_\Lambda{}^\Sigma{}_\Gamma$, while the nonvanishing left-lower block $T_{\Lambda\Sigma\Gamma}$ is symmetric in Σ and Γ.

Furthermore we note that (2.25) implies

$$F_\Lambda(X)\,\delta X^\Lambda = -g\Lambda^\Gamma T_{\Gamma\Sigma}{}^\Lambda\,F_\Lambda(X)\,X^\Sigma = -\frac{1}{2}g\,\Lambda^\Lambda\,T_{\Lambda\Sigma\Gamma}X^\Sigma X^\Gamma\,. \tag{2.30}$$

Upon replacing Λ^Λ with X^Λ we conclude that the fully symmetric part of $T_{\Lambda\Sigma\Gamma}$ vanishes. This, and the closure of the gauge group, leads to the following three equations,

$$\begin{aligned} T_{(\Lambda\Sigma\Gamma)} &= 0\,, \\ T_{[\Lambda\Sigma}{}^\Delta\,T_{\Gamma]\Delta}{}^\Xi &= 0\,, \\ 4\,T_{(\Gamma[\Lambda}{}^\Delta\,T_{\Sigma]\Xi)\Delta} - T_{\Lambda\Sigma}{}^\Delta T_{\Delta\Gamma\Xi} &= 0\,. \end{aligned} \tag{2.31}$$

JHEP08(2007)064

The variation of the Lagrangian (2.26) under gauge transformations now takes the form

$$\mathcal{L} \to \mathcal{L} + \frac{1}{8}\mathrm{i}\,\varepsilon^{\mu\nu\rho\sigma}\,\Lambda^\Lambda\,T_{\Lambda\Sigma\Gamma}\,\mathcal{F}_{\mu\nu}{}^\Sigma\mathcal{F}_{\rho\sigma}{}^\Gamma\,, \tag{2.32}$$

where the tensors $\mathcal{F}_{\mu\nu}{}^\Lambda$ denote the non-abelian field strengths,

$$\mathcal{F}_{\mu\nu}{}^\Lambda = \partial_\mu A_\nu{}^\Lambda - \partial_\nu A_\mu{}^\Lambda + g\,T_{\Sigma\Gamma}{}^\Lambda\,A_\mu{}^\Sigma A_\nu{}^\Gamma\,. \tag{2.33}$$

This result implies that (2.32) no longer constitutes a total derivative in view of the space-time dependent transformation parameters $\Lambda^\Lambda(x)$. Therefore its cancellation requires to add a new type of term [8],

$$\mathcal{L} = \frac{1}{3}\mathrm{i}g\,\varepsilon^{\mu\nu\rho\sigma}\,T_{\Lambda\Sigma\Gamma}\,A_\mu{}^\Lambda A_\nu{}^\Sigma\Big(\partial_\rho A_\sigma{}^\Gamma + \frac{3}{8}g\,T_{\Xi\Delta}{}^\Gamma\,A_\rho{}^\Xi A_\sigma{}^\Delta\Big)\,. \tag{2.34}$$

No other terms in the action will depend on $T_{\Lambda\Sigma\Gamma}$. At this point we should remind the reader that the gauging breaks supersymmetry, unless one adds the standard masslike and potential terms to the Lagrangian (2.1), which involve the $T_{\Lambda\Sigma}{}^\Gamma$. We present them below for completeness,

$$\begin{aligned}
\mathcal{L}_g &= -\frac{1}{2}g\,N_{\Lambda\Sigma}T_{\Gamma\Xi}{}^\Sigma\Big[\varepsilon^{ij}\,\bar\Omega_i{}^\Lambda\Omega_j{}^\Gamma\bar X^\Xi + \varepsilon_{ij}\,\bar\Omega^{i\Lambda}\Omega^{j\Gamma}X^\Xi\Big]\,,\\
\mathcal{L}_{g^2} &= g^2\,N_{\Lambda\Sigma}\,T_{\Gamma\Xi}{}^\Lambda\bar X^\Gamma X^\Xi\,T_{\Delta\Omega}{}^\Sigma\bar X^\Delta X^\Omega\,.
\end{aligned} \tag{2.35}$$

In later sections we will exhibit the generalization of these terms to the case where both electric and magnetic charges are present.

2.2 Electric and magnetic charges

We now consider more general gauge groups without restricting ourselves to electric charges. Therefore we include both electric gauge fields $A_\mu{}^\Lambda$ and magnetic gauge fields $A_{\mu\,\Lambda}$. Only a subset of these fields is usually involved in the gauging, but the additional magnetic gauge fields could conceivably lead to new propagating degrees of freedom. We will discuss in due course how this is avoided. In this subsection we consider the scalar and spinor fields. The treatment of the vector fields is more involved and is explained in section 3.

The charges $T_{MN}{}^P$ correspond to a more general subgroup of the duality group. Hence they must take values in the Lie algebra associated with $\mathrm{Sp}(2n,\mathbb{R})$, which implies,

$$T_{M[N}{}^Q\,\Omega_{P]Q} = 0\,. \tag{2.36}$$

Combining the two equations (2.16) and (2.23) leads to the condition [10],

$$T_{MN}{}^Q\Omega_{PQ}\,X^NX^P = T_{M\Lambda\Sigma}X^\Lambda X^\Sigma - 2T_{M\Lambda}{}^\Sigma X^\Lambda F_\Sigma - T_M{}^{\Lambda\Sigma}F_\Lambda F_\Sigma = 0\,. \tag{2.37}$$

This result can also be written as

$$F_\Lambda\delta X^\Lambda = -\frac{1}{2}\Lambda^M\Big(T_{M\Lambda\Sigma}X^\Lambda X^\Sigma + T_M{}^{\Lambda\Sigma}F_\Lambda F_\Sigma\Big)\,, \tag{2.38}$$

JHEP08(2007)064

which generalizes (2.30). Furthermore we impose the so-called representation constraint [1], which implies that we suppress a representation of the rigid symmetry group in $T_{MN}{}^{P}$,

$$T_{(MN}{}^{Q}\,\Omega_{P)Q} = 0 \Longrightarrow \begin{cases} T^{(\Lambda\Sigma\Gamma)} = 0\,, \\ 2T^{(\Gamma\Lambda)}{}_{\Sigma} = T_{\Sigma}{}^{\Lambda\Gamma}\,, \\ T_{(\Lambda\Sigma\Gamma)} = 0\,, \\ 2T_{(\Gamma\Lambda)}{}^{\Sigma} = T^{\Sigma}{}_{\Lambda\Gamma}\,. \end{cases} \tag{2.39}$$

This constraint is a generalization of the first equation (2.31). Observe that the generators $T_{\Lambda\Sigma}{}^{\Gamma}$ are no longer antisymmetric in Λ and Σ, a feature that we will discuss in more detail in section 3.

The action of electric/magnetic duality on the fermions was already discussed earlier when introducing the $\mathrm{Sp}(2n,\mathbb{R})$ covariant fermionic vector $\Omega_i{}^M$ (c.f. (2.19)). In terms of this field we can rewrite the Lagrangian (2.2) in a compact form,

$$\mathcal{L}_{\text{matter}} = -\mathrm{i}\Omega_{MN}\,\partial_\mu X^M\,\partial^\mu \bar{X}^N + \frac{1}{4}\mathrm{i}\Omega_{MN}\Big[\bar{\Omega}^{iM}\partial\!\!\!/\Omega_i{}^N - \bar{\Omega}_i{}^M\partial\!\!\!/\Omega^{iN}\Big]\,. \tag{2.40}$$

In the expressions on the right-hand side it is straightforward to replace the ordinary derivatives by the covariant ones defined in (2.28), i.e.,

$$\begin{aligned} D_\mu X^M &= \partial_\mu X^M + g\,A_\mu{}^N\,T_{NP}{}^M\,X^P\,, \\ D_\mu \Omega_i{}^M &= \partial_\mu \Omega_i{}^M + g\,A_\mu{}^N\,T_{NP}{}^M\,\Omega_i{}^P\,, \end{aligned} \tag{2.41}$$

and evaluate the gauge couplings. In particular we can then compare to the results of subsection 2.1, where we considered only electric gauge fields with charges restricted by $T_\Lambda{}^{\Sigma\Gamma} = 0$. To do this systematically we note the identity,

$$T_{MN\Lambda}X^N - F_{\Lambda\Sigma}\,T_{MN}{}^{\Sigma}X^N = 0\,. \tag{2.42}$$

This equation can also be written as $F_{\Lambda\Sigma}\,\delta X^\Sigma = -g\Lambda^M T_{MN\Lambda}X^N$, which is the infinitesimal form of the first equation (2.24). Alternatively it can be derived from (2.37) upon differentiation with respect to X^Λ.

It is possible to cast (2.42) in a symplectically covariant form by introducing a vector $U^M = (U^\Lambda, F_{\Sigma\Gamma}U^\Gamma)$, so that

$$\Omega_{MQ}T_{NP}{}^{Q}\,X^P\,U^M = 0\,, \tag{2.43}$$

for any such vector U^M. This form is convenient in calculations presented later.

From (2.42) one easily derives that $D_\mu X_\Lambda = D_\mu F_\Lambda = F_{\Lambda\Sigma}\,D_\mu X^\Sigma$, which enables one to derive

$$-\,\mathrm{i}\Omega_{MN}\,D_\mu X^M\,D^\mu \bar{X}^N = -N_{\Lambda\Sigma}\,D_\mu X^\Lambda\,D^\mu \bar{X}^\Sigma\,. \tag{2.44}$$

This result shows that the generators $T_{M\Lambda\Sigma}$ are absent, in accord with what was found in subsection 2.1.

Next we consider the gauge field interactions with the fermions. It is convenient to first derive an additional identity, which follows from taking a supersymmetry variation of (2.42),

$$T_{MN\Lambda}\Omega_i{}^N = F_{\Lambda\Sigma}\,T_{MN}{}^{\Sigma}\Omega_i{}^N + F_{\Lambda\Sigma\Gamma}\,\Omega_i{}^\Sigma\,T_{MN}{}^{\Gamma}X^N\,. \tag{2.45}$$

This result can be obtained from the infinitesimal form of the third equation of (2.24). Using this equation one verifies that $D_\mu\Omega_{i\Lambda} = F_{\Lambda\Sigma}\, D_\mu\Omega_i{}^\Sigma + F_{\Lambda\Sigma\Gamma}\,\Omega_i{}^\Gamma D_\mu X^\Sigma$, which leads to

$$\frac{1}{4}\mathrm{i}\Omega_{MN}\Big[\bar\Omega^{iM}\not{D}\Omega_i{}^N - \bar\Omega_i{}^M\not{D}\Omega^{iN}\Big] = -\frac{1}{4}N_{\Lambda\Sigma}\Big(\bar\Omega^{i\Lambda}\not{D}\Omega_i{}^\Sigma + \bar\Omega_i{}^\Lambda\not{D}\Omega^{i\Sigma}\Big)$$
$$-\frac{1}{4}\mathrm{i}\Big(F_{\Lambda\Sigma\Gamma}\bar\Omega_i{}^\Lambda\not{D}X^\Sigma\Omega^{i\Gamma} - \bar F_{\Lambda\Sigma\Gamma}\bar\Omega^{i\Lambda}\not{D}\bar X^\Sigma\Omega_i{}^\Gamma\Big)\,. \quad (2.46)$$

Again the generator $T_{M\Lambda\Sigma}$ is absent in the expression above. The results of this subsection explain how to introduce the electric and magnetic charges, but in no way ensure the gauge invariance or the supersymmetry of the Lagrangian. To obtain such a result we first need to explain some more general features of theories with both electric and magnetic gauge fields in four space-time dimensions. This is the topic of the following section.

As a side remark we note that the Killing potential (or moment map) associated with the isometries considered above, takes the form,

$$\nu_M = T_{MN}{}^Q\Omega_{PQ}\bar X^N X^P\,. \quad (2.47)$$

Indeed, making use again of (2.42), one straightforwardly derives $\partial_\Lambda\nu_M = \mathrm{i}N_{\Lambda\Sigma}\,\delta\bar X^\Sigma$.

Finally we return to the gauge transformations of the auxiliary fields $Y_{ij}{}^\Lambda$, which can be derived by requiring that the Lagrangian (2.5) is gauge invariant. A straightforward calculation lead to the following result,

$$\delta Y_{ij}{}^\Lambda = -\frac{1}{2}\Lambda^M T_{MN}{}^\Lambda(Z_{ij}{}^N + \varepsilon_{ik}\varepsilon_{jl}\,Z^{klN})\,, \quad (2.48)$$

where $Z_{ij}{}^M$ was defined in (2.22). Note that this result is in accord with the electric/magnetic dualities suggested for $Z_{ij}{}^M$.

3. The gauge group and the embedding tensor

Here we follow [1] and discuss the embedding of possible gauge groups into the rigid invariance group $\mathrm{G}_{\mathrm{rigid}}$ of the theory. In the context of this paper, the latter is often a product group as the vector multiplets and the hypermultiplets are invariant under independent symmetry groups. As explained in the previous section the non-abelian gauge transformations on the vector multiplets must be embedded into the electric/magnetic duality group.

It is convenient to discuss group embeddings in terms of a so-called embedding tensor $\Theta_M{}^{\mathsf{a}}$ which specifies the decomposition of the gauge group generators T_M into the generators associated with the full rigid invariance group $\mathrm{G}_{\mathrm{rigid}}$,

$$T_M = \Theta_M{}^{\mathsf{a}}\,t_{\mathsf{a}}\,. \quad (3.1)$$

Not all the gauge fields have to be involved in the gauging, so generically the embedding tensor projects out certain combinations of gauge fields; the rank of the tensor determines the dimension of the gauge group, up to central extensions associated with abelian factors.

JHEP08(2007)064

Decomposing the embedding tensor as $\Theta_M{}^{\mathsf{a}} = (\Theta_\Lambda{}^{\mathsf{a}}, \Theta^{\Lambda\,\mathsf{a}})$, covariant derivatives take the form,

$$D_\mu \equiv \partial_\mu - g A_\mu{}^M T_M = \partial_\mu - g A_\mu{}^\Lambda \Theta_\Lambda{}^{\mathsf{a}}\, t_{\mathsf{a}} - g A_{\mu\,\Lambda} \Theta^{\Lambda\,\mathsf{a}}\, t_{\mathsf{a}} \; . \tag{3.2}$$

The embedding tensor will be regarded as a spurionic object which can be assigned to a (not necessarily irreducible) representation of the rigid invariance group $\mathrm{G}_{\mathrm{rigid}}$.

It is known that a number of ($\mathrm{G}_{\mathrm{rigid}}$-covariant) constraints must be imposed on the embedding tensor. We already encountered the representation constraint (2.39), which is linear in the embedding tensor. Two other constraints are quadratic in the embedding tensor and read,

$$f_{\mathsf{ab}}{}^{\mathsf{c}}\, \Theta_M{}^{\mathsf{a}}\, \Theta_N{}^{\mathsf{b}} + (t_{\mathsf{a}})_N{}^P\, \Theta_M{}^{\mathsf{a}} \Theta_P{}^{\mathsf{c}} = 0\,, \tag{3.3}$$

$$\Omega^{MN}\, \Theta_M{}^{\mathsf{a}} \Theta_N{}^{\mathsf{b}} \;=\; 0 \quad\Longleftrightarrow\quad \Theta^{\Lambda\,[\mathsf{a}} \Theta_\Lambda{}^{\mathsf{b}]} = 0\,, \tag{3.4}$$

where the $f_{\mathsf{ab}}{}^{\mathsf{c}}$ are the structure constants associated with the group G. The first constraint is required by the closure of the gauge group generators. Indeed, from (3.3) it follows that the gauge algebra generators close according to

$$[T_M, T_N] = -T_{MN}{}^P\, T_P \; , \tag{3.5}$$

where the structure constants of the gauge group coincide with $T_{MN}{}^P \equiv \Theta_M{}^{\mathsf{a}}\, (t_{\mathsf{a}})_N{}^P$ up to terms that vanish upon contraction with the embedding tensor $\Theta_P{}^{\mathsf{a}}$. We recall that the $T_{MN}{}^P$ generate a subgroup of $\mathrm{Sp}(2n,\mathbb{R})$ in the $(2n)$-dimensional representation, so that they are subject to the condition (2.36). In electric/magnetic components the latter condition corresponds to $T_{M\Lambda}{}^\Sigma = -T_M{}^\Sigma{}_\Lambda$, $T_{M\Lambda\Sigma} = T_{M\Sigma\Lambda}$ and $T_M{}^{\Lambda\Sigma} = T_M{}^{\Sigma\Lambda}$.

Note that (3.3) implies that the embedding tensor is gauge invariant, while the second quadratic constraint (3.4) implies that the charges are mutually local, so that an electric/magnetic duality exists that converts all the charges to electric ones. These two quadratic constraints are not completely independent, as can be seen from symmetrizing the constraint (3.3) in (MN) and making use of the linear conditions (2.39) and (2.36). This leads to

$$\Omega^{MN}\, \Theta_M{}^{\mathsf{a}} \Theta_N{}^{\mathsf{b}}\, (t_{\mathsf{b}})_P{}^Q = 0\; . \tag{3.6}$$

This shows that, for non-vanishing $(t_{\mathsf{b}})_P{}^Q$, the second quadratic constraint (3.4) is in fact a consequence of the other constraints. The constraint (3.4) is only an independent constraint when a and b do not refer to generators that act on the vector multiplets. This issue is relevant here as $\mathrm{G}_{\mathrm{rigid}}$ may contain independent generators that act exclusively in the matter (i.e., hypermultiplet) sector.

A further consequence of (2.39) is the equation

$$T_{(MN)}{}^P = Z^{P,\mathsf{a}}\, d_{\mathsf{a}\,MN} \; , \tag{3.7}$$

with

$$\begin{aligned} d_{\mathsf{a}\,MN} &\equiv (t_{\mathsf{a}})_M{}^P\, \Omega_{NP}\,, \\ Z^{M,\mathsf{a}} &\equiv \frac{1}{2}\Omega^{MN}\Theta_N{}^{\mathsf{a}} \quad\Longrightarrow\quad \left\{ \begin{aligned} Z^{\Lambda\mathsf{a}} &= \tfrac{1}{2}\Theta^{\Lambda\mathsf{a}}\,, \\ Z_\Lambda{}^{\mathsf{a}} &= -\tfrac{1}{2}\Theta_\Lambda{}^{\mathsf{a}}\,, \end{aligned} \right. \end{aligned} \tag{3.8}$$

JHEP08(2007)064

so that $d_{\mathsf{a}\,MN}$ defines a $\mathrm{G}_{\rm rigid}$-invariant tensor symmetric in (MN). The gauge invariant tensor $Z^{M,\mathsf{a}}$ will serve as a projector on the tensor fields to be introduced below [16]. We note that the constraint (3.4) can now be written as,

$$Z^{M,\mathsf{a}}\,\Theta_M{}^{\mathsf{b}} = 0\,. \tag{3.9}$$

Let us return to the closure relation (3.5). Although the left-hand side is antisymmetric in M and N, this does not imply that $T_{MN}{}^P$ is antisymmetric as well, but only that its symmetric part vanishes upon contraction with the embedding tensor. Indeed, this is reflected by (3.7) and (3.9). Consequently, the Jacobi identity holds only modulo terms that vanish upon contraction with the embedding tensor, as is shown explicitly by

$$T_{[MN]}{}^P\,T_{[QP]}{}^R + T_{[QM]}{}^P\,T_{[NP]}{}^R + T_{[NQ]}{}^P\,T_{[MP]}{}^R = -Z^{R,\mathsf{a}}\,d_{\mathsf{a}\,P[Q}\,T_{MN]}{}^P\,. \tag{3.10}$$

To compensate for this lack of closure and, at the same time, to avoid unwanted degrees of freedom, we introduce an extra gauge invariance for the gauge fields, in addition to the usual nonabelian gauge transformations,

$$\delta A_\mu{}^M = D_\mu\Lambda^M - g\,Z^{M,\mathsf{a}}\,\Xi_{\mu\,\mathsf{a}}\,, \tag{3.11}$$

where the Λ^M are the gauge transformation parameters and the covariant derivative reads, $D_\mu\Lambda^M = \partial_\mu\Lambda^M + g\,T_{PQ}{}^M\,A_\mu{}^P\Lambda^Q$. The transformations proportional to $\Xi_{\mu\,\mathsf{a}}$ enable one to gauge away those vector fields that are in the sector of the gauge generators $T_{MN}{}^P$ where the Jacobi identity is not satisfied (this sector is perpendicular to the embedding tensor by virtue of (3.9)). Note that the covariant derivative is invariant under the transformations parametrized by $\Xi_{\mu\,\mathsf{a}}$, because of the contraction of the gauge fields $A_\mu{}^M$ with the generators T_M. The gauge symmetries parametrized by the functions $\Lambda^M(x)$ and $\Xi_{\mathsf{a}\mu}(x)$ form a group, as follows from the commutation relations,

$$\begin{aligned}
[\delta(\Lambda_1),\delta(\Lambda_2)] &= \delta(\Lambda_3) + \delta(\Xi_3)\,,\\
[\delta(\Lambda),\delta(\Xi)] &= \delta(\tilde\Xi)\,,
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
\Lambda_3{}^M &= g\,T_{[NP]}{}^M\Lambda_2^N\Lambda_1^P\,,\\
\Xi_{3\mu\mathsf{a}} &= d_{\mathsf{a}NP}(\Lambda_1^N D_\mu\Lambda_2^P - \Lambda_2^N D_\mu\Lambda_1^P)\,,\\
\tilde\Xi_{\mu\mathsf{a}} &= g\Lambda^P(T_{P\mathsf{a}}{}^{\mathsf{b}} + 2d_{\mathsf{a}PN}Z^{N,\mathsf{b}})\Xi_{\mu\mathsf{b}}\,.
\end{aligned} \tag{3.13}$$

The field strengths follow from the Ricci identity, $[D_\mu, D_\nu] = -g\mathcal{F}_{\mu\nu}{}^M\,T_M$, and depend only on the antisymmetric part of $T_{MN}{}^P$,

$$\mathcal{F}_{\mu\nu}{}^M = \partial_\mu A_\nu{}^M - \partial_\nu A_\mu{}^M + g\,T_{[NP]}{}^M\,A_\mu{}^N A_\nu{}^P\,. \tag{3.14}$$

Because of the lack of closure expressed by (3.10), they do not satisfy the Palatini identity,

$$\delta\mathcal{F}_{\mu\nu}{}^M = 2\,D_{[\mu}\delta A_{\nu]}{}^M - 2g\,T_{(PQ)}{}^M\,A_{[\mu}{}^P\,\delta A_{\nu]}{}^Q\,, \tag{3.15}$$

JHEP08(2007)064

under arbitrary variations $\delta A_\mu{}^M$. Note that the last term cancels upon multiplication with the generators T_M. The result (3.15) shows that $\mathcal{F}_{\mu\nu}{}^M$ transforms under gauge transformations as

$$\delta\mathcal{F}_{\mu\nu}{}^M = g\,\Lambda^P T_{NP}{}^M\,\mathcal{F}_{\mu\nu}{}^N - 2g\,Z^{M,\mathsf{a}}(D_{[\mu}\Xi_{\nu]\mathsf{a}} + d_{\mathsf{a}PQ}\,A_{[\mu}{}^P\,\delta A_{\nu]}{}^Q)\,, \tag{3.16}$$

and is therefore not covariant. The standard strategy is therefore to define modified field strengths,

$$\mathcal{H}_{\mu\nu}{}^M = \mathcal{F}_{\mu\nu}{}^M + g\,Z^{M,\mathsf{a}}\,B_{\mu\nu\,\mathsf{a}}\,, \tag{3.17}$$

by introducing new tensor fields $B_{\mu\nu\,\mathsf{a}}$ with suitably chosen gauge transformation rules, so that covariant results can be obtained.

At this point we remind the reader that the invariance transformations in the rigid case implied that the field strengths $G_{\mu\nu}{}^M$ transform under a subgroup of $\mathrm{Sp}(2n,\mathbb{R})$ (c.f. (2.13)). Our aim is to find a similar symplectric vector of field strengths so that these transformations are generated in the non-abelian case as well. This is not possible based on the variations of the vector fields $A_\mu{}^M$, which will never generate the type of fermionic terms contained in $G_{\mu\nu\Lambda}$. However, the presence of the tensor fields enables us to achieve our objectives, at least in part. Just as in the abelian case, we define an $\mathrm{Sp}(2n,\mathbb{R})$ vector of field strengths $\mathcal{G}_{\mu\nu}{}^M$ by

$$\begin{aligned}
\mathcal{G}^{-}_{\mu\nu}{}^{\Lambda} &= \mathcal{H}^{-}_{\mu\nu}{}^{\Lambda}\,,\\
\mathcal{G}^{-}_{\mu\nu\Lambda} &= F_{\Lambda\Sigma}\,\mathcal{H}^{-}_{\mu\nu}{}^{\Sigma} - \frac{1}{8}F_{\Lambda\Sigma\Gamma}\,\bar\Omega_i{}^{\Sigma}\gamma_{\mu\nu}\Omega_j{}^{\Gamma}\,\varepsilon^{ij}\,.
\end{aligned} \tag{3.18}$$

Note that the expression for $\mathcal{G}_{\mu\nu\Lambda}$ is the analogue of (2.11), with $F_{\mu\nu}{}^\Lambda$ replaced by $\mathcal{H}_{\mu\nu}{}^\Lambda$.

Following [1] we introduce the following transformation rule for $B_{\mu\nu\mathsf{a}}$ (contracted with $Z^{M,\mathsf{a}}$, because only these combinations will appear in the Lagrangian),

$$Z^{M,\mathsf{a}}\,\delta B_{\mu\nu\,\mathsf{a}} = 2\,Z^{M,\mathsf{a}}(D_{[\mu}\Xi_{\nu]\mathsf{a}} + d_{\mathsf{a}\,NP}A_{[\mu}{}^N\delta A_{\nu]}{}^P) - 2\,T_{(NP)}{}^M\Lambda^P\mathcal{G}_{\mu\nu}{}^N\,, \tag{3.19}$$

where $D_\mu\Xi_{\nu\mathsf{a}} = \partial_\mu\Xi_{\nu\mathsf{a}} - gA_\mu{}^M T_{M\mathsf{a}}{}^{\mathsf{b}}\Xi_{\nu\mathsf{b}}$ with $T_{M\mathsf{a}}{}^{\mathsf{b}} = -\Theta_M{}^{\mathsf{c}}f_{\mathsf{ca}}{}^{\mathsf{b}}$ the gauge group generator in the adjoint representation of $\mathrm{G}_{\mathrm{rigid}}$. With this variation the modified field strengths (3.17) are invariant under tensor gauge transformations. Under the vector gauge transformations we derive the following result,

$$\begin{aligned}
\delta\mathcal{G}^{-}_{\mu\nu}{}^{\Lambda} &= -g\,\Lambda^P T_{PN}{}^{\Lambda}\,\mathcal{G}^{-}_{\mu\nu}{}^N - g\,\Lambda^P T^{\Gamma}{}_P{}^{\Lambda}\,(\mathcal{G}^{-}_{\mu\nu} - \mathcal{H}^{-}_{\mu\nu})_\Gamma\,,\\
\delta\mathcal{G}^{-}_{\mu\nu\Lambda} &= -g\,\Lambda^P T_{PN\Lambda}\,\mathcal{G}^{-}_{\mu\nu}{}^N - g\,F_{\Lambda\Sigma}\,\Lambda^P T^{\Gamma}{}_P{}^{\Sigma}\,(\mathcal{G}^{-}_{\mu\nu} - \mathcal{H}^{-}_{\mu\nu})_\Gamma\,,\\
\delta(\mathcal{G}^{-}_{\mu\nu} - \mathcal{H}^{-}_{\mu\nu})_\Lambda &= g\,\Lambda^P(T^{\Gamma}{}_{P\Lambda} - T^{\Gamma}{}_P{}^{\Sigma}\,F_{\Sigma\Lambda})\,(\mathcal{G}^{-}_{\mu\nu} - \mathcal{H}^{-}_{\mu\nu})_\Gamma\,.
\end{aligned} \tag{3.20}$$

Hence $\delta\mathcal{G}_{\mu\nu}{}^M = -g\,\Lambda^P T_{PN}{}^M\,\mathcal{G}^N_{\mu\nu}$, just as the variation of the abelian field strengths $G_{\mu\nu}{}^M$ in the absence of charges, up to terms proportional to $\Theta^{\Lambda,\mathsf{a}}(\mathcal{G}_{\mu\nu} - \mathcal{H}_{\mu\nu})_\Lambda$. According to [1], the latter terms represent a set of field equations. In that case the last equation of (3.20) expresses the well-known fact that, under a symmetry, field equations transform into field equations. As a result the gauge algebra on these tensors closes according to (3.12), up to the same field equations.

In order that the Lagrangian (2.3) becomes invariant under the vector and tensor gauge transformations, we have to make a number of changes. First of all, we replace the abelian field strengths $F_{\mu\nu}{}^{\Lambda}$ in (2.3) by $\mathcal{H}_{\mu\nu}{}^{\Lambda}$, so that

$$\mathcal{G}_{\mu\nu\,\Lambda} = \mathrm{i}\,\varepsilon_{\mu\nu\rho\sigma}\,\frac{\partial\mathcal{L}_{\rm vector}}{\partial\mathcal{H}_{\rho\sigma}{}^{\Lambda}}\,. \tag{3.21}$$

Under general variations of the vector and tensor fields we then obtain the result,

$$\delta\mathcal{L}_{\rm vector} = -\mathrm{i}\mathcal{G}^{+\mu\nu}{}_{\Lambda}\Big[D_\mu\delta A_\nu{}^{\Lambda} + \frac{1}{4}g\Theta^{\Lambda\mathsf{a}}(\delta B_{\mu\nu\mathsf{a}} - 2d_{\mathsf{a}PQ}A_\mu{}^P\delta A_\nu{}^Q)\Big] + \text{h.c.}\,. \tag{3.22}$$

The reader can check that the Lagrangian (2.3) is indeed invariant under the tensor gauge transformations. Even when we include the transformations of the scalar and spinor fields, the Lagrangian is, however, not yet invariant under the vector gauge transformations. For that it is necessary to introduce the following universal terms to the Lagrangian [1],

$$\begin{aligned}\mathcal{L}_{\rm top} = {}& \frac{1}{8}\mathrm{i}g\,\varepsilon^{\mu\nu\rho\sigma}\,\Theta^{\Lambda\mathsf{a}}\,B_{\mu\nu\,\mathsf{a}}\Big(2\,\partial_\rho A_{\sigma\,\Lambda} + gT_{MN\,\Lambda}\,A_\rho{}^M A_\sigma{}^N - \frac{1}{4}g\Theta_\Lambda{}^{\mathsf{b}}B_{\rho\sigma\,\mathsf{b}}\Big)\\ &+\frac{1}{3}\mathrm{i}g\,\varepsilon^{\mu\nu\rho\sigma}T_{MN\,\Lambda}\,A_\mu{}^M A_\nu{}^N\Big(\partial_\rho A_\sigma{}^{\Lambda} + \frac{1}{4}gT_{PQ}{}^{\Lambda}A_\rho{}^P A_\sigma{}^Q\Big)\\ &+\frac{1}{6}\mathrm{i}g\,\varepsilon^{\mu\nu\rho\sigma}T_{MN}{}^{\Lambda}\,A_\mu{}^M A_\nu{}^N\Big(\partial_\rho A_{\sigma\Lambda} + \frac{1}{4}gT_{PQ\Lambda}A_\rho{}^P A_\sigma{}^Q\Big)\,.\end{aligned} \tag{3.23}$$

The first term represents a topological coupling of the antisymmetric tensor fields with the magnetic gauge fields, and the last two terms are a generalization of the Chern-Simons-like terms (2.34) that we encountered in subsection 2.1. Under variations of the vector and tensor fields, this Lagrangian varies into (up to total derivative terms)

$$\delta\mathcal{L}_{\rm top} = \mathrm{i}\mathcal{H}^{+\mu\nu\Lambda}\,D_\mu\delta A_{\nu\Lambda} + \frac{1}{4}\mathrm{i}g\,\mathcal{H}^{+\mu\nu}{}_{\Lambda}\,\Theta^{\Lambda\mathsf{a}}(\delta B_{\mu\nu\mathsf{a}} - 2d_{\mathsf{a}PQ}A_\mu{}^P\delta A_\nu{}^Q) + \text{h.c.}\,. \tag{3.24}$$

Under the tensor gauge transformations this variation becomes equal to $(\mathrm{i}g\,\mathcal{H}^{+\mu\nu M}\,\Theta_M{}^{\mathsf{a}}\,D_\mu\Xi_{\nu\mathsf{a}} + \text{h.c.})$. This expression equals a total derivative by virtue of (3.9) and the Bianchi identity,

$$D_{[\mu}\mathcal{H}_{\nu\rho]}{}^M = \frac{1}{3}g\,Z^{M,\mathsf{a}}\,\mathcal{H}_{\mu\nu\rho\mathsf{a}}\,, \tag{3.25}$$

where

$$\mathcal{H}_{\mu\nu\rho\mathsf{a}} \equiv 3\,D_{[\mu}B_{\nu\rho]\,\alpha} + 6\,d_{\alpha\,NP}\,A_{[\mu}{}^N(\partial_\nu A_{\rho]}{}^P + \frac{1}{3}gT_{[RS]}{}^P A_\nu{}^R A_{\rho]}{}^S)\,. \tag{3.26}$$

In the above equations, covariant derivatives are defined by $D_\mu\mathcal{H}_{\nu\rho}{}^M = \partial_\mu\mathcal{H}_{\nu\rho}{}^M + gA_\mu{}^P T_{PN}{}^M\mathcal{H}_{\nu\rho}{}^N$ and $D_\rho B_{\mu\nu\mathsf{a}} = \partial_\rho B_{\mu\nu\alpha} - gA_\rho{}^M T_{M\mathsf{a}}{}^{\mathsf{b}}B_{\mu\nu\mathsf{b}}$. Observe that these derivatives are not fully covariant in view of (3.20) and (3.19). Fully covariantized expressions were presented in [3] but are not needed below. The gauge invariance of the total Lagrangian $\mathcal{L}_{\rm vector} + \mathcal{L}_{\rm top}$, will follow upon including the gauge transformations of the matter fields [1].

As we stressed before, the combined gauge invariance of the vector and tensor gauge fields ensures that the number of physical degrees of freedom will not change by the introduction of the magnetic vector gauge fields and the tensor gauge fields [1]. The combined gauge algebra is consistent for the tensor fields upon projection with the embedding tensor, and as it turns out the action depends only on those field components. If this were not the case, one would need to introduce new tensor fields of higher rank [15, 16]. Indeed, under variation of the tensor fields one finds

$$\delta\mathcal{L}_{\text{vector}} + \delta\mathcal{L}_{\text{top}} = -\frac{1}{8}\mathrm{i}g\,\varepsilon^{\mu\nu\rho\sigma}\,(\mathcal{G}-\mathcal{H})_{\mu\nu\Lambda}\,\Theta^{\Lambda\mathsf{a}}\,\delta B_{\rho\sigma\mathsf{a}}\,, \tag{3.27}$$

which shows that the components of the tensor fields that are projected to zero by multiplication with $\Theta^{\Lambda\mathsf{a}}$ are not present in the action. Hence those components can be associated with an additional gauge invariance. A similar situation arises with the magnetic gauge fields $A_{\mu\Lambda}$. Under variations of the gauge fields ${A_\mu}^M$ one derives,

$$\delta\mathcal{L}_{\text{vector}} + \delta\mathcal{L}_{\text{top}} = \frac{1}{2}\mathrm{i}\,\varepsilon^{\mu\nu\rho\sigma}\,D_\nu{\mathcal{G}_{\rho\sigma}}^M\Omega_{MN}\delta{A_\mu}^N\,, \tag{3.28}$$

up to a total derivative and up to terms that vanish as a result of the field equation for $B_{\mu\nu\alpha}$. Substituting (3.25) we rewrite (3.28) as follows,

$$\delta\mathcal{L}_{\text{vector}} + \delta\mathcal{L}_{\text{top}} = \frac{1}{2}\mathrm{i}\,\varepsilon^{\mu\nu\rho\sigma}\left[-D_\nu\mathcal{G}_{\rho\sigma\Lambda}\,\delta{A_\mu}^\Lambda + \frac{1}{6}g\,\mathcal{H}_{\nu\rho\sigma\mathsf{a}}\,\Theta^{\Lambda\mathsf{a}}\delta A_{\mu\Lambda}\right]. \tag{3.29}$$

Because the minimal coupling of the gauge fields is always proportional to the embedding tensor, the full Lagrangian does not change under variations of the magnetic gauge fields that are projected to zero by the embedding tensor component $\Theta^{\Lambda\mathsf{a}}$, up to terms that are generated by the variations of the tensor fields through the 'universal' variation, $\delta B_{\mu\nu\mathsf{a}} = 2\,d_{\mathsf{a}PQ}{A_\mu}^P\delta{A_\nu}^Q$.

Finally, we have been able to identify yet another independent gauge invariance which acts only on the tensor fields,

$$\Theta^{\Lambda\mathsf{a}}\delta B_{\mu\nu\mathsf{a}} \propto {\Delta^{\Lambda\Sigma\rho}}_\rho\,(\mathcal{G}-\mathcal{H})_{\mu\nu\Sigma} - 6\,{\Delta^{(\Lambda\Sigma)\rho}}_{[\rho}\,(\mathcal{G}-\mathcal{H})_{\mu\nu]\Sigma}\,, \tag{3.30}$$

where ${\Delta^{\Lambda\Sigma\mu}}_\nu = \Theta^{\Lambda\mathsf{a}}{\Delta_\mathsf{a}}^{\Sigma\mu}{}_\nu$.

All these gauge symmetries have a role to play in balancing the degrees of freedom. In [1] a precise accounting of all gauge symmetries was bypassed in the analysis. We note that not all of them have a bearing on the dynamical modes of the theory as they also act on fields that play an auxiliary role.

4. Restoring supersymmetry for non-abelian vector multiplets

In this section we show how the supersymmetry can be restored in the presence of a gauging. In this way we will find the generalizations of the masslike and potential terms of order g and g^2, respectively, which were already exhibited in (2.35) for the case of purely electric charges. In addition we determine the corresponding changes in the transformation rules. The supersymmetry transformations that leave the action corresponding to (2.1) invariant, were given in (2.8).

Introducing electric and magnetic charges, with a uniform gauge coupling constant g as before, requires a number of universal changes of the Lagrangian that were already discussed in the previous section. In $\mathcal{L}_{\rm matter}$ we have to covariantize the derivatives as already discussed in subsection 2.2. It is convenient to use the representation (2.40). With the covariantizations included we thus have

$$\mathcal{L}_{\rm matter} = -\mathrm{i}\Omega_{MN}\, D_\mu X^M\, D^\mu \bar{X}^N + \frac{1}{4}\mathrm{i}\Omega_{MN}\Big[\bar{\Omega}^{iM}\,\not{D}\Omega_i{}^N - \bar{\Omega}_i{}^M\,\not{D}\Omega^{iN}\Big]\,. \tag{4.1}$$

In $\mathcal{L}_{\rm vector}$ we must replace the abelian field strengths $F_{\mu\nu}{}^\Lambda$ by the modified field strengths $\mathcal{H}_{\mu\nu}{}^\Lambda$, defined in (3.17). Therefore we replace (2.3) by

$$\begin{aligned}\mathcal{L}_{\rm vector} = {}& \frac{1}{4}\mathrm{i}F_{\Lambda\Sigma}\mathcal{H}^{-\Lambda}_{\mu\nu}\mathcal{H}^{-\Sigma\,\mu\nu} - \frac{1}{16}\mathrm{i}F_{\Lambda\Sigma\Gamma}\bar{\Omega}^{\Lambda}_i\,\gamma^{\mu\nu}\mathcal{H}^{-\Sigma}_{\mu\nu}\,\Omega^{\Gamma}_j\,\varepsilon^{ij}\\ & -\frac{1}{256}\mathrm{i}N^{\Delta\Omega}\Big(F_{\Delta\Lambda\Sigma}\bar{\Omega}_i{}^\Lambda\gamma_{\mu\nu}\Omega_j{}^\Sigma\varepsilon^{ij}\Big)\Big(F_{\Gamma\Xi\Omega}\bar{\Omega}_k{}^\Gamma\gamma^{\mu\nu}\Omega_l{}^\Xi\varepsilon^{kl}\Big) + \text{h.c.}\,. \end{aligned}\tag{4.2}$$

Furthermore one includes the Lagrangians (2.4), (2.5) and (3.23), which remain unaltered. Up to an extension of (2.35), whose form we will establish in this section, we do not expect further modifications.

Also the supersymmetry transformation rules acquire a number of modifications, extending space-time derivatives and field strengths to covariant ones. Furthermore one has to take account of the presence of the new magnetic gauge fields and the tensor fields. However, one also needs a few additional terms in the transformation rules, whose form will be established in due course. For the moment we use the following modified transformation rules, where we also include the variations of the magnetic gauge fields, which we denote by δ_0,

$$\begin{aligned}\delta_0 X^\Lambda &= \bar{\epsilon}^i\Omega_i{}^\Lambda\,,\\ \delta_0 A_\mu{}^\Lambda &= \varepsilon^{ij}\bar{\epsilon}_i\gamma_\mu\Omega_j{}^\Lambda + \varepsilon_{ij}\bar{\epsilon}^i\gamma_\mu\Omega^{j\Lambda}\,,\\ \delta_0 A_{\mu\Lambda} &= F_{\Lambda\Sigma}\,\varepsilon^{ij}\bar{\epsilon}_i\gamma_\mu\Omega_j{}^\Sigma + \bar{F}_{\Lambda\Sigma}\,\varepsilon_{ij}\bar{\epsilon}^i\gamma_\mu\Omega^{j\Sigma}\,,\\ \delta_0\Omega_i{}^\Lambda &= 2\not{D}X^\Lambda\epsilon_i + \frac{1}{2}\gamma^{\mu\nu}\mathcal{H}^{-}_{\mu\nu}{}^\Lambda\varepsilon_{ij}\epsilon^j + Y_{ij}{}^\Lambda\epsilon^j\,,\\ \delta_0 Y_{ij}{}^\Lambda &= 2\bar{\epsilon}_{(i}\not{D}\Omega_{j)}{}^\Lambda + 2\varepsilon_{ik}\varepsilon_{jl}\,\bar{\epsilon}^{(k}\not{D}\Omega^{l)\Lambda}\,. \end{aligned}\tag{4.3}$$

At this point it is convenient to note that the supersymmetry variations of the scalar, spinor and vector fields can be written in the form,

$$\begin{aligned}\delta_0 X^M &= \bar{\epsilon}^i\Omega_i{}^M\,,\\ \delta_0 A_\mu{}^M &= \varepsilon^{ij}\bar{\epsilon}_i\gamma_\mu\Omega_j{}^M + \varepsilon_{ij}\bar{\epsilon}^i\gamma_\mu\Omega^{jM}\,,\\ \delta_0\Omega_i{}^M &= 2\not{D}X^M\epsilon_i + \frac{1}{2}\gamma^{\mu\nu}\mathcal{G}^{-}_{\mu\nu}{}^M\varepsilon_{ij}\epsilon^j + Z_{ij}{}^M\epsilon^j\,, \end{aligned}\tag{4.4}$$

where the fermions $\Omega_i{}^M$, the field strengths $\mathcal{G}_{\mu\nu}{}^M$, and the quantities $Z_{ij}{}^M$ were defined in (2.19), (3.18) and (2.22), respectively.

Most of the cancellations required for demonstrating the supersymmetry of the Lagrangian will still take place when derivatives are replaced by covariant derivatives. A clear exception arises when dealing with the commutator of two derivatives, because they will lead to field strengths upon using the Ricci identity. This situation arises for the variations of the fermion kinetic term. Furthermore, when establishing supersymmetry for the more conventional Lagrangians, one makes use of the Bianchi identity for the field strengths, which no longer applies to the new field strenghts. Of course, the presence of gauge fields in the covariant derivatives induces new variations. To investigate these issues, we first determine the supersymmetry variation of $\mathcal{L}_{\rm matter}$ under the transformations given above (up to total derivatives),

$$\begin{aligned}\delta_0\mathcal{L}_{\rm matter} = {}& {\rm i}g\,\Omega_{MQ}T_{PN}{}^{Q}\left[D^\mu\bar{X}^M\,X^N-\bar{X}^M\,D_\mu X^N+\frac{1}{2}\bar{\Omega}^{iM}\gamma^\mu\Omega_i{}^N\right]\delta A_\mu{}^P\\ &-\frac{1}{2}{\rm i}g\,\Omega_{MQ}T_{PN}{}^{Q}\,\left[\bar{X}^M\,\bar{\Omega}_i{}^N\gamma^{\mu\nu}\epsilon^i\,\mathcal{H}^-_{\mu\nu}{}^P-{\rm h.c.}\right]\\ &+{\rm i}\,\Omega_{MN}\,\left[\bar{\Omega}^{iM}\gamma_\nu\epsilon^j\,\varepsilon_{ij}\,D_\mu\mathcal{G}^{-\mu\nu N}-{\rm h.c.}\right]\,,\end{aligned}\tag{4.5}$$

where we suppressed variations that involve neither the gauge coupling constant g nor the (modified) field strengths. These variations will cancel as before.

It is now easy to verify that the term of order g^0 can be combined with the result from the variation of $\mathcal{L}_{\rm vector}+\mathcal{L}_{\rm top}$ (c.f. (3.22) and (3.24)),

$$\delta_0(\mathcal{L}_{\rm vector}+\mathcal{L}_{\rm top})=-{\rm i}\Omega_{MN}\,\mathcal{G}^{-\mu\nu M}\,D_\mu\delta A_\nu{}^N+{\rm h.c.}+\cdots\,.\tag{4.6}$$

Upon using the expressions for $\mathcal{G}_{\mu\nu\Lambda}$ and $\delta A_{\mu\Lambda}$, the combined result thus leads to a total derivative plus terms proportional to $D_\mu F_{\Lambda\Sigma}$ and terms cubic in the fermions. These terms cancel for the abelian theory with an ordinary derivative and the cancellation proceeds identically when ordinary derivatives are replaced by covariant ones. Note that nowhere one needs to use the Bianchi identity. This calculation confirms the correctness of the transformation rule for the magnetic gauge fields. Hence we can now concentrate on the remaining terms of (4.5), which are the only variations left, up to terms induced by the variation of the tensor fields which we will need in due course.

To cancel the order-g terms in (4.5) we need to add new terms in the transformation rules of $\Omega_i{}^\Lambda$ and $Y_{ij}{}^\Lambda$. Furthermore new terms to the Lagrangian are required. For the case of purely electric charges these terms are known and the obvious strategy is to simply generalize these terms. This leads to the expressions,

$$\begin{aligned}\delta_g\Omega_i{}^\Lambda &= -2g\,T_{MN}{}^\Lambda\,\bar{X}^M X^N\,\varepsilon_{ij}\,\epsilon^j\,,\\ \delta_g Y_{ij}{}^\Lambda &= -4g\,T_{MN}{}^\Lambda\left[\bar{\Omega}_{(i}{}^M\epsilon^k\,\varepsilon_{j)k}\,\bar{X}^N-\bar{\Omega}^{kM}\epsilon_{(i}\,\varepsilon_{j)k}\,X^N\right]\,,\\ \mathcal{L}_g &= -\frac{1}{2}{\rm i}g\,\Omega_{MQ}T_{PN}{}^Q\,\left[\varepsilon^{ij}\,\bar{\Omega}_i{}^M\Omega_j{}^P\bar{X}^N-\varepsilon_{ij}\,\bar{\Omega}^{iM}\Omega^{jP}X^N\right]\,.\end{aligned}\tag{4.7}$$

In the case of purely electric charges the expression for $\mathcal{L}_g$ reduces to the first expression of (2.35) upon using (2.42).

Collecting the new variations proportional to the field strengths that arise as a result of (4.7), we find, using (3.18), (2.45) and (2.39),

$$\delta_g\mathcal{L}_{\rm vector}+\delta_0\mathcal{L}_g=\frac{1}{2}{\rm i}g\,\Omega_{MQ}T_{PN}{}^Q\,\bar{X}^M\,\bar{\Omega}_i{}^N\gamma^{\mu\nu}\epsilon^i\,\mathcal{G}^-_{\mu\nu}{}^P+{\rm h.c.}\,.\tag{4.8}$$

This term is almost identical to the second term of (4.5) except that is proportional to $\mathcal{G}_{\mu\nu}{}^M$ rather than to $\mathcal{H}_{\mu\nu}{}^M$. However, the combination of these two terms is cancelled by assigning the following variation to the tensor fields,

$$\delta B_{\mu\nu\,\mathsf{a}} = -2t_{\mathsf{a}M}{}^P\Omega_{PN}\left(A_{[\mu}{}^M\,\delta A_{\nu]}{}^N - \bar{X}^M\bar{\Omega}_i{}^N\gamma_{\mu\nu}\epsilon^i - X^M\bar{\Omega}^{iN}\gamma_{\mu\nu}\epsilon_i\right)\,. \tag{4.9}$$

At this point one can verify that all other supersymmetry variations linear in the gauge coupling constant g vanish. Here one makes use of the various results derived in subsection 2.2, and in particular of (2.43). What remains are the order-g^2 interactions induced by the order-g transformations of the spinors, which can be written as,

$$\delta_g\Omega_i{}^M = -2g\,T_{NP}{}^M\,\bar{X}^N X^P\,\varepsilon_{ij}\,\epsilon^j\,. \tag{4.10}$$

The order-g^2 variation follows from $\delta_g\mathcal{L}_g$, and can be written proportional to the supersymmetry variation δX^M given in (4.4),

$$\delta_g\mathcal{L}_g = -2\mathrm{i}g^2\,\Omega_{MQ}T_{NP}{}^Q\,\bar{X}^P\delta X^{[M}\,T_{RS}{}^{N]}\,\bar{X}^R X^S + \text{h.c.}\,. \tag{4.11}$$

Using the Lie algebra relation (3.5), as well as the relation (2.43), we can write this in a form that can be integrated. This reveals that these variations can be cancelled by the variation of a scalar potential, corresponding to

$$\mathcal{L}_{g^2} = \mathrm{i}g^2\,\Omega_{MN}\,T_{PQ}{}^M X^P\bar{X}^Q\,T_{RS}{}^N\bar{X}^R X^S\,. \tag{4.12}$$

This expression reduces to (2.35) for purely electric gaugings upon using (2.42). Observe that the charges $T_{\Lambda\Sigma\Gamma}$ do not contribute to (4.12), as is well known from previous constructions.

Before closing this section we determine the supersymmetry algebra by evaluating the supersymmetry commutator on X^M and $A_\mu{}^M$ (bearing in mind that the magnetic gauge fields $A_{\mu\Lambda}$ can be contracted with $\Theta^{\Lambda\mathsf{a}}$ without loss of generality). The result for the commuatator takes the following form,

$$[\delta(\epsilon_1),\delta(\epsilon_2)] = 2(\bar{\epsilon}_2{}^i\gamma^\mu\epsilon_{1i} + \bar{\epsilon}_{2i}\gamma^\mu\epsilon_1{}^i)D_\mu + \delta(\Lambda) + \delta(\Xi)\,, \tag{4.13}$$

where the first term corresponds to a covariant translation (covariant with respect to vector and tensor gauge transformations), and the second and third terms denote additional vector and tensor gauge transformations with parameters,

$$\begin{aligned}\Lambda^M &= 4\,(\bar{X}^M\,\bar{\epsilon}_2{}^i\epsilon_1{}^j\,\varepsilon_{ij} + X^M\,\bar{\epsilon}_{2i}\epsilon_{1j}\,\varepsilon^{ij})\,,\\ \Xi_{\mu\mathsf{a}} &= -2\,d_{\mathsf{a}NP}(A_\mu{}^N A_\nu{}^P + 2\,\eta_{\mu\nu}\bar{X}^N X^P)(\bar{\epsilon}_2{}^i\gamma^\nu\epsilon_{1i} + \bar{\epsilon}_{2i}\gamma^\nu\epsilon_1{}^i)\,.\end{aligned} \tag{4.14}$$

Here we made use of (2.37) to close the commutator on X^M. For closing the commutator on $A_\mu{}^M$ we used the field equations for $Y_{ij}{}^M$ (implying that $Z_{ij}{}^M$ is pseudo-real), and the field equation for $B_{\mu\nu\mathsf{a}}$.

This concludes the derivation of supersymmetric vector multiplet Lagrangians with electric and magnetic gauge charges. In the following section we will consider the coupling to matter by introducing hypermultiplets. This will lead to additional contributions to the scalar potential.

JHEP08(2007)064

5. Hypermultiplets

In this section we give a brief description of hypermultiplets and their gaugings, following the framework of [17, 18]. The $n_{\rm H}$ hypermultiplets are described by $4n_{\rm H}$ real scalars ϕ^A, $2n_{\rm H}$ positive-chirality spinors $\zeta^{\bar\alpha}$ and $2n_{\rm H}$ negative-chirality spinors ζ^α. Hence target-space indices $A, B, \ldots$ take values $1, 2, \ldots, 4n_{\rm H}$, and the indices $\alpha, \beta, \ldots$ and $\bar\alpha, \bar\beta, \ldots$ run from 1 to $2n_{\rm H}$. The chiral and antichiral spinors are related by complex conjugation (so that we have $2n_{\rm H}$ Majorana spinors) under which indices are converted according to $\alpha \leftrightarrow \bar\alpha$.

The supersymmetry transformations take the form,

$$\begin{aligned}
\delta_0\phi^A &= 2(\gamma^A_{i\bar\alpha}\,\bar\epsilon^i\zeta^{\bar\alpha} + \bar\gamma^{Ai}_\alpha\,\bar\epsilon_i\zeta^\alpha)\,,\\
\delta_0\zeta^\alpha &= V^\alpha_{A\,i}\,\partial\!\!\!/\phi^A\epsilon^i - \delta\phi^A\,\Gamma_A{}^\alpha{}_\beta\,\zeta^\beta\,,\\
\delta_0\zeta^{\bar\alpha} &= \bar V^{i\bar\alpha}_A\,\partial\!\!\!/\phi^A\epsilon_i - \delta\phi^A\,\bar\Gamma_A{}^{\bar\alpha}{}_{\bar\beta}\,\zeta^{\bar\beta}\,,
\end{aligned}\tag{5.1}$$

where δ_0 indicates that the variations refer to zero gauge coupling constant g. Here $\Gamma_A{}^\alpha{}_\beta$ and $\Gamma_A{}^{\bar\alpha}{}_{\bar\beta}$ are the connections associated with field-dependent reparametrizations of the fermions of the form $\zeta^\alpha \to S^\alpha{}_\beta(\phi)\,\zeta^\beta$, and $\zeta^{\bar\alpha} \to \bar S^{\bar\alpha}{}_{\bar\beta}(\phi)\,\zeta^{\bar\beta}$. Naturally these reparametrizations act on all quantities carrying indices α and $\bar\alpha$. The curvatures $R_{AB}{}^\alpha{}_\beta$ and $R_{AB}{}^{\bar\alpha}{}_{\bar\beta}$ associated with these connections take their values in $\mathrm{sp}(n_{\rm H}) \cong \mathrm{usp}(2n_{\rm H};\mathbb{C})$. The quantities γ^A and V_A are $(4n_{\rm H})\times(4n_{\rm H})$ complex matrices which play the role of the quaternionic (inverse) vielbeine of the target space. They satisfy a pseudo-reality condition specified below.

The Lagrangian takes the following form

$$\mathcal{L}_0 = -\frac{1}{2}g_{AB}\,\partial_\mu\phi^A\partial^\mu\phi^B - G_{\bar\alpha\beta}(\bar\zeta^{\bar\alpha}D\!\!\!\!/\,\zeta^\beta + \bar\zeta^\beta D\!\!\!\!/\,\zeta^{\bar\alpha}) - \frac{1}{4}W_{\bar\alpha\beta\bar\gamma\delta}\,\bar\zeta^{\bar\alpha}\gamma_\mu\zeta^\beta\,\bar\zeta^{\bar\gamma}\gamma^\mu\zeta^\delta\,, \tag{5.2}$$

with covariant derivatives

$$D_\mu\zeta^\alpha = \partial_\mu\zeta^\alpha + \partial_\mu\phi^A\,\Gamma_A{}^\alpha{}_\beta\,\zeta^\beta\,,\quad D_\mu\zeta^{\bar\alpha} = \partial_\mu\zeta^{\bar\alpha} + \partial_\mu\phi^A\,\bar\Gamma_A{}^{\bar\alpha}{}_{\bar\beta}\,\zeta^{\bar\beta}\,. \tag{5.3}$$

The tensor $W_{\bar\alpha\beta\bar\gamma\delta}$ is related to the Riemann curvature R_{ABCD} associated with the target space metric g_{AB}, as well as to the $\mathrm{sp}(n_{\rm H})$ curvatures mentioned above. Observe that the Lagrangian is invariant under the U(1) R-symmetry group which acts by chiral transformations on the fermion fields. The SU(2) R-symmetry can only be realized when the target space has an SU(2) isometry.

The target-space metric g_{AB}, the tensors γ^A, V_A and the fermionic hermitean metric $G_{\bar\alpha\beta}$ (i.e., satisfying $(G_{\bar\alpha\beta})^* = G_{\bar\beta\alpha}$) are all covariantly constant with respect to the Christoffel connection and the connections $\Gamma_A{}^\alpha{}_\beta$ and $\Gamma_A{}^{\bar\alpha}{}_{\bar\beta}$. Furthermore we note the following relations,

$$\begin{aligned}
\bar\gamma^j_{A\alpha}\,V^\alpha_{Bi} &= \gamma_{Bi\bar\alpha}\,\bar V^{j\bar\alpha}_A = -\bar\gamma^j_{B\alpha}\,V^\alpha_{iA} + \delta^j_i\,g_{AB}\,,\\
\bar V^{i\bar\alpha}_A\,\gamma^A_{j\bar\beta} &= \delta^i_j\,\delta^{\bar\alpha}_{\bar\beta}\,,\\
g^{AB}\,V^\alpha_{Ai}\,V^\beta_{Bj} &= \varepsilon_{ij}\,\Omega^{\alpha\beta}\,,\qquad g_{AB}\,\gamma^A_{i\bar\alpha}\,\gamma^B_{j\bar\beta} = \varepsilon_{ij}\,\Omega_{\bar\alpha\bar\beta}\,,\\
\varepsilon_{ij}\,\Omega_{\bar\alpha\bar\beta}\,\bar V^{j\bar\beta}_A &= g_{AB}\,\gamma^B_{i\bar\alpha} = G_{\bar\alpha\beta}\,V^\beta_{A\,i}\,,\\
\gamma_{Ai\bar\alpha}\,\bar V^{j\bar\alpha}_B &= \varepsilon_{ik}J^{kj}_{AB} + \frac{1}{2}g_{AB}\,\delta^j_i\,,\\
J_{AB}{}^{ij}\gamma^B_{\bar\alpha k} &= -\delta^{(i}_k\,\varepsilon^{j)l}\,\gamma_{A\bar\alpha l}\,.
\end{aligned}\tag{5.4}$$

Here $\Omega^{\alpha\beta}$ and $\Omega_{\bar\alpha\bar\beta}$ are skew-symmetric covariantly constant tensors (satisfying $\Omega_{\bar\alpha\bar\beta}\bar\Omega^{\bar\beta\bar\gamma} = -\delta_{\bar\alpha}{}^{\bar\gamma}$), and the J^{ij}_{AB} are three complex structures generating the algebra of quaternions. The existence of the complex structures implies that the target space is hyperkähler.

The equivalence transformations of the fermions and the target-space diffeomorphisms do not constitute invariances of the theory, unless they leave the metric g_{AB} and the $\mathrm{Sp}(n_{\mathrm{H}}) \times \mathrm{Sp}(1)$ one-form V_i^α (and thus the related geometric quantities) invariant. Therefore invariances are related to isometries of the hyperkähler space. A subset of them can be elevated to a group of local (i.e. space-time-dependent) transformations, which require a coupling to corresponding vector multiplets. Such gauged isometries have been studied in the literature [19 – 24] but only for electric charges.

Infinitesimal isometries are characterized by Killing vectors and the ones associated to local transformations will be labeled by the same index M that labels the electric and magnetic gauge fields of the previous sections. In principle, the gauged isometries constitute a subgroup of the full group of isometries, defined by the embedding tensor. Hence the corresponding Killing vectors are proportional to the embedding matrix, $k^A{}_M = \Theta_M{}^{\mathsf{a}}\, k^A{}_{\mathsf{a}}$, and (3.9) implies,

$$Z^{M,\mathsf{a}}\, k^A{}_M = 0\,. \tag{5.5}$$

Without gauge interactions, the hypermultiplets do not couple to the vector multiplets, so that the full group of invariances factorizes into separate invariance groups of the vector multiplet Lagrangian and of the hypermultiplet Lagrangian. The index a refers to all these symmetries, and therefore $k^A{}_{\mathsf{a}}$ will vanish whenever the index a refers to a generator acting exclusively on the vector multiplets.

The local gauge group is thus generated by the Killing vectors $k^A{}_M(\phi) = (k^A{}_\Lambda(\phi), k^{A\Lambda}(\phi))$, with parameters Λ^M. Under infinitesimal transformations we have

$$\delta\phi^A = g\,\Lambda^M k^A{}_M(\phi)\,, \tag{5.6}$$

where g is the coupling constant and the $k^A{}_M(\phi)$ satisfy the Killing equation,

$$D_A k_{BM} + D_B k_{AM} = 0\,. \tag{5.7}$$

Higher derivatives of Killing vectors are not independent, as is shown by

$$D_A D_B k_{CM} = R_{BCAE}\, k^E{}_M\ . \tag{5.8}$$

The isometries close under commutation,

$$k^B{}_M \partial_B k^A{}_N - k^B{}_N \partial_B k^A{}_M = T_{MN}{}^P\, k^A{}_P\,, \tag{5.9}$$

where, as before, the antisymmetry in $[MN]$ on the right-hand side is ensured by (5.5).

The invariances associated with the target space isometries act on the fermions by field dependent matrices, which satify the relation

$$(t_M)^\alpha{}_\beta\, V^\beta_{Ai} = D_A k^B{}_M\, V^\alpha_{Bi}\,, \tag{5.10}$$

leading to

$$(t_M)^\alpha{}_\beta = \frac{1}{2} V^\alpha_{Ai}\, \bar\gamma^{Bi}_\beta\; D_B k^A{}_M \,. \tag{5.11}$$

The result (5.10) was derived by requiring that the tensor V^α_{Ai} is invariant under the isometries, up to a rotation on the indices α. The invariance implies that target-space scalars satisfy algebraic identities such as

$$\bar t_M{}^{\bar\gamma}{}_{\bar\alpha}\, G_{\bar\gamma\beta} + t_M{}^\gamma{}_\beta\, G_{\bar\alpha\gamma} = t_M{}^{\bar\gamma}{}_{[\bar\alpha}\, \Omega_{\bar\beta]\bar\gamma} = 0\,, \tag{5.12}$$

which establishes that the matrices $t_M{}^\alpha{}_\beta$ take values in $\mathrm{sp}(n_\mathrm{H})$. From (5.9) and (5.8), one may derive

$$D_A t_M{}^\alpha{}_\beta = R_{AB}{}^\alpha{}_\beta\, k^B{}_M \,, \tag{5.13}$$

for any infinitesimal isometry. From the group property of the isometries it follows that the matrices t_M satisfy the commutation relations,

$$[\, t_M ,\, t_N\,]^\alpha{}_\beta = -T_{MN}{}^P\, (t_P)^\alpha{}_\beta + k^A{}_M\, k^B{}_N\, R_{AB}{}^\alpha{}_\beta \,, \tag{5.14}$$

which takes values in $\mathrm{sp}(n_\mathrm{H})$. This result is consistent with the Jacobi identity.

The previous results imply that the complex structures J^{ij}_{AB} are invariant under the isometries,

$$k^C{}_M\, \partial_C J^{ij}_{AB} - 2\partial_{[A} k^C{}_M\, J^{ij}_{B]C} = 0\,, \tag{5.15}$$

implying that the isometries are *tri-holomorphic*. From (5.15) one shows that $\partial_A (J^{ij}_{BC}\, k^C{}_M) - \partial_B (J^{ij}_{AC}\, k^C{}_M) = 0$, so that, locally, one can associate three Killing potentials (or moment maps) $\mu^{ij}{}_M$ to every Killing vector, according to

$$\partial_A \mu^{ij}{}_M = J^{ij}_{AB}\, k^B{}_M \,, \tag{5.16}$$

which determines $\mu^{ij}{}_M$ up to a constant. These constants correspond to Fayet-Iliopoulos terms. Up to such constants one derives the equivariance condition,

$$J^{ij}_{AB}\, k^A{}_M\, k^B{}_N = T_{MN}{}^P\, \mu^{ij}{}_P \,, \tag{5.17}$$

which implies that the Killing potentials transform covariantly under the isometries,

$$\delta\mu^{ij}{}_M = \Lambda^N\, k^A{}_N\, \partial_A \mu^{ij}_M = \Lambda^N\, T_{NM}{}^P\, \mu^{ij}{}_P \,. \tag{5.18}$$

Subsequently we consider the consequences of realizing the isometry (sub)group generated by the $k^A{}_M$ as a local gauge group. The latter acts on the hypermultiplet fields in the following way,

$$\delta\phi = g\, \Lambda^M\, k^A{}_M \,, \qquad \delta\zeta^\alpha = g\, \Lambda^M t_M{}^\alpha{}_\beta\, \zeta^\beta - \delta\phi^A \Gamma_A{}^\alpha{}_\beta\, \zeta^\beta \,, \tag{5.19}$$

where the parameters Λ^M are functions of x^μ. The relevant covariant derivatives are equal to,

$$\mathcal{D}_\mu \phi^A = \partial_\mu \phi^A - g A_\mu{}^M\, k^A{}_M \,, \qquad \mathcal{D}_\mu \zeta^\alpha = \partial_\mu \zeta^\alpha + \partial_\mu \phi^A\, \Gamma_A{}^\alpha{}_\beta\, \zeta^\beta - g A_\mu{}^M t_M{}^\alpha{}_\beta\, \zeta^\beta \,. \tag{5.20}$$

These covariant derivatives must be substituted into the transformation rules (5.1) and the Lagrangian (5.2). The covariance of $\mathcal{D}_\mu\zeta^\alpha$,

$$\delta\mathcal{D}_\mu\zeta^\alpha = g\,\Lambda^M t_M{}^\alpha{}_\beta\,\mathcal{D}_\mu\zeta^\beta - \delta\phi^A\Gamma_A{}^\alpha{}_\beta\,\mathcal{D}_\mu\zeta^\beta\,. \tag{5.21}$$

follows from (5.13) and (5.14).

Just as for the vector multiplets, the introduction of the gauge covariant derivatives to the Lagrangian breaks the supersymmetry of the Lagrangian. To restore supersymmetry we follow the same procedure as in section 4. But in this case the situation is somewhat simpler because the electric and magnetic gauge fields couple to standard hypermultiplet isometries. This means that the initial results will coincide with those obtained for electric gaugings.

Let us first present the variations of the Lagrangian (5.2) with the proper gauge covariantizations and determine the supersymmetry variation linear in the gauge coupling constant g and linear in the fermion fields,

$$\delta\mathcal{L}_0 = g\,k_{AM}\Big[\gamma^A_{i\bar\alpha}\,\bar\zeta^{\bar\alpha}\gamma^{\mu\nu}\epsilon^i\mathcal{F}^{-\,M}_{\mu\nu} + \varepsilon^{ij}\,\bar\Omega_i{}^M\not{\mathcal{D}}\phi^A\epsilon_j + \text{h.c.}\Big]\,. \tag{5.22}$$

The first term originates from the fact that the commutator of two covariant derivatives acquires an extra field strength in the presence of the gauging, whereas the second term originates from the variation of the gauge fields in the covariant derivatives of the scalars. The first term can be cancelled by a supersymmetry variation of the following new term,

$$\mathcal{L}_g^{(1)} = 2g\,k_{AM}\left[\bar\gamma^{Ai}_\alpha\varepsilon_{ij}\,\bar\zeta^\alpha\Omega^{jM} + \gamma^A_{i\bar\alpha}\varepsilon^{ij}\,\bar\zeta^{\bar\alpha}\Omega_j{}^M\right]\,. \tag{5.23}$$

The variations of this term proportional to the field strength $\mathcal{G}_{\mu\nu}{}^M$ cancel against the term proportional to $\mathcal{H}_{\mu\nu}{}^M$ (the field strength $\mathcal{F}_{\mu\nu}{}^M$ can be replaced by $\mathcal{H}_{\mu\nu}{}^M$ by virtue of (5.5)) by adding a new term to the variation (4.9) of the tensor fields $B_{\mu\nu\mathsf{a}}$,

$$\delta B_{\mu\nu\mathsf{a}} = -4\mathrm{i}k^A{}_{\mathsf{a}}\,\left[\gamma_{Ai\bar\alpha}\,\bar\zeta^{\bar\alpha}\gamma_{\mu\nu}\epsilon^i - \bar\gamma^i_{A\alpha}\,\bar\zeta^\alpha\gamma_{\mu\nu}\epsilon_i\right]\,. \tag{5.24}$$

Another term in the variation of (5.23) is proportional to X^M and its complex conjugate. Their cancellation requires the following extra variations of the hypermultiplet spinors,

$$\delta\zeta^\alpha = 2gX^M\,k^A{}_M V^\alpha_{Ai}\,\varepsilon^{ij}\epsilon_j\,, \qquad \delta\zeta^{\bar\alpha} = 2g\bar X^M\,k^A{}_M\bar V^{\bar\alpha i}_A\,\varepsilon_{ij}\epsilon^j\,, \tag{5.25}$$

and an extra term in the Lagrangian equal to

$$\mathcal{L}_g^{(2)} = 2g\,\Big[\bar X^M t_M{}^\gamma{}_\alpha\,\bar\Omega_{\beta\gamma}\,\bar\zeta^\alpha\zeta^\beta + X^M t_M{}^{\bar\gamma}{}_{\bar\alpha}\,\Omega_{\bar\beta\bar\gamma}\,\bar\zeta^{\bar\alpha}\zeta^{\bar\beta}\Big]\,. \tag{5.26}$$

The remaining variations then take the following form.

$$\begin{aligned}\delta\mathcal{L}_0 + \delta\mathcal{L}_g^{(1)} + \delta\mathcal{L}_g^{(2)} = &-2g\,\partial_A\mu^{ij}{}_M\,\bar\Omega_i{}^M\not{\mathcal{D}}\phi^A\epsilon_j - 2g\,\partial_A\mu_{ijM}\,\bar\Omega^{iM}\not{\mathcal{D}}\phi^A\epsilon^j\\ &-2g\,\left[\partial_A\mu_{ij\Lambda}\,Y^{ij\Lambda} + \partial_A\mu_{ij}{}^\Lambda\,\bar F_{\Lambda\Sigma}\,Y^{ij\Sigma}\right]\,\bar\gamma^{Ak}_\alpha\,\bar\epsilon_k\zeta^\alpha\\ &-2g\,\left[\partial_A\mu_{ij\Lambda}\,Y^{ij\Lambda} + \partial_A\mu_{ij}{}^\Lambda\,F_{\Lambda\Sigma}\,Y^{ij\Sigma}\right]\,\gamma^A_{k\bar\alpha}\,\bar\epsilon^k\zeta^{\bar\alpha}\,,\end{aligned} \tag{5.27}$$

where we restricted ourselves to variations linear in the fermion fields and linear in g.

To cancel these variations we must include the following new term to the Lagrangian,

$$\mathcal{L}_g^{(3)} = g\, Y^{ij\Lambda}\Big[\mu_{ij\Lambda} + \frac{1}{2}(F_{\Lambda\Sigma} + \bar{F}_{\Lambda\Sigma})\,\mu_{ij}{}^{\Sigma}\Big] - \frac{1}{4}g\,\big[F_{\Lambda\Sigma\Gamma}\,\mu^{ij\Lambda}\,\bar{\Omega}_i{}^{\Sigma}\Omega_j{}^{\Gamma} + \bar{F}_{\Lambda\Sigma\Gamma}\,\mu_{ij}{}^{\Lambda}\,\bar{\Omega}^{i\Sigma}\Omega^{j\Gamma}\big]\,, \tag{5.28}$$

as well as assign new variations of the fields $\Omega_i{}^{\Lambda}$ and Y_{ij}^{Λ} of the vector multiplet,

$$\begin{aligned}
\delta_g\Omega_i{}^{\Lambda} &= 2\,\mathrm{i}g\,\mu_{ij}{}^{\Lambda}\epsilon^j\,,\\
\delta_g Y_{ij}{}^{\Lambda} &= 4\,\mathrm{i}g\,k^{A\Lambda}\Big[\varepsilon_{k(i}\,\gamma_{j)\bar{\alpha}A}\bar{\epsilon}^k\zeta^{\bar{\alpha}} + \varepsilon_{k(i}\,\bar{\epsilon}_{j)}\zeta^{\alpha}\,\bar{\gamma}^k_{\alpha A}\Big]\,.
\end{aligned} \tag{5.29}$$

This completes the discussion of all the variations linear in g and in the fermion fields. The result remains valid for the cubic fermion variations as well. However, new variations arise in second order in g, by the order-g variations in the order-g terms in the Lagrangian. Here we have to consider the combined results for the vector multiplets and the hypermultiplets. All these variations cancel against the variation of a scalar potential, corresponding to

$$\mathcal{L}_{g^2} = -2g^2 k^A{}_M\, k^B{}_N\, g_{AB}\, X^M\bar{X}^N - \frac{1}{2}g^2\, N_{\Lambda\Sigma}\,\mu_{ij}{}^{\Lambda}\,\mu^{ij\Sigma}\,. \tag{5.30}$$

6. Off-shell structure

In the absence of magnetic charges, the vector multiplets constitute off-shell representations of the $N = 2$ supersymmetry algebra and the tensor fields decouple from the theory. However, on the hypermultiplets the supersymmetry algebra is only realized up to fermionic field equations. The situation changes crucially when magnetic charges are present. In that case there are no longer any off-shell multiplets and the supersymmetry algebra is only realized when the fields satisfy the field equations of the hypermultiplet spinors and of the fields $A_{\mu\Lambda}$, $Y_{ij}{}^{\Lambda}$ and $B_{\mu\nu\mathsf{a}}$. In this section we discuss how the off-shell closure can be regained for the vector multiplets when magnetic charges are switched on. In this discussion the hypermultiplet fields play only an ancillary role.

We start by introducing $2n$ *independent* vector multiplets, associated with the electric and magnetic gauge fields, $A_\mu{}^{\Lambda}$ and $A_{\mu\Lambda}$, and collectively denoted by $A_\mu{}^M$. In the absence of charges, these fields are subject to the standard off-shell transformation rules,

$$\begin{aligned}
\delta X^M &= \bar{\epsilon}^i\Omega_i{}^M\,,\\
\delta A_\mu{}^M &= \varepsilon^{ij}\bar{\epsilon}_i\gamma_\mu\Omega_j{}^M + \varepsilon_{ij}\bar{\epsilon}^i\gamma_\mu\Omega^{jM}\,,\\
\delta\Omega_i{}^M &= 2\partial\!\!\!/ X^M\epsilon_i + \frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}^{-\,M}\varepsilon_{ij}\epsilon^j + Y_{ij}{}^M\epsilon^j\,,\\
\delta Y_{ij}{}^M &= 2\bar{\epsilon}_{(i}\partial\!\!\!/\Omega_{j)}{}^M + 2\varepsilon_{ik}\varepsilon_{jl}\,\bar{\epsilon}^{(k}\partial\!\!\!/\Omega^{l)M}\,.
\end{aligned} \tag{6.1}$$

We stress once more that, unlike previously, these $2n$ vector multiplets are independent. In due course we shall see how to make contact with the previous description.

The tensor gauge fields $B_{\mu\nu\mathsf{a}}$ are assigned to off-shell tensor multiplets. Just as before, the index a labels the independent continuous symmetries of the theory. These multiplets consist of scalar fields $L^{ij}{}_{\mathsf{a}}$, positive chirality spinors $\varphi^i{}_{\mathsf{a}}$ (and their negative chirality

JHEP08(2007)064

conjugates $\varphi_{i\mathsf{a}}$), tensor gauge fields $B_{\mu\nu\mathsf{a}}$, and complex scalars G_a. However, for reasons explained below, we complexify the scalars $L^{ij}{}_\mathsf{a}$ by introducing *complex* scalars $P^{ij}{}_\mathsf{a}$. These fields transform as vectors under the SU(2) R-symmetry, and their pseudo-real parts are proportional to the fields $L^{ij}{}_\mathsf{a}$,

$$L^{ij}{}_\mathsf{a} = P^{ij}{}_\mathsf{a} + \varepsilon^{ik}\varepsilon^{jl} P_{kl\mathsf{a}} \,. \tag{6.2}$$

The consistency of this extension is ensured by introducing, at the same time, the local gauge transformations, $P^{ij}{}_\mathsf{a}(x) \to P^{ij}{}_\mathsf{a}(x) + \mathrm{i}\xi^{ij}{}_\mathsf{a}(x)$, where the gauge parameters $\xi^{ij}{}_\mathsf{a}$ are pseudo-real, so that $\xi_{ij\mathsf{a}} = \varepsilon_{ik}\varepsilon_{jl}\,\xi^{kl}{}_\mathsf{a}$. In terms of the gauge invariant scalars $L^{ij}{}_\mathsf{a}$ we will obtain the more conventional formulation of the tensor multiplet.[2] The supersymmetry variations of the tensor multiplets are now as follows,

$$\begin{aligned}
\delta P_{ij\mathsf{a}} &= 2\,\varepsilon_{ik}\varepsilon_{jl}\,\bar\epsilon^{(k}\varphi^{l)}{}_\mathsf{a}\,,\\
\delta B_{\mu\nu\mathsf{a}} &= \frac{1}{2}\mathrm{i}\bar\epsilon^i\gamma_{\mu\nu}\varphi^j{}_\mathsf{a}\,\varepsilon_{ij} - \frac{1}{2}\mathrm{i}\bar\epsilon_i\gamma_{\mu\nu}\varphi_{j\mathsf{a}}\,\varepsilon^{ij}\,,\\
\delta\varphi^i{}_\mathsf{a} &= \partial\!\!\!/(P^{ij}{}_\mathsf{a} + \varepsilon^{ik}\varepsilon^{jl}P_{kl\mathsf{a}})\,\epsilon_j + 2\,\varepsilon^{ij}\,H\!\!\!/_\mathsf{a}\epsilon_j - G_\mathsf{a}\epsilon^i\,,\\
\delta G_\mathsf{a} &= -2\,\bar\epsilon_i\partial\!\!\!/\varphi^i{}_\mathsf{a}\,,
\end{aligned} \tag{6.3}$$

where $H^\mu{}_\mathsf{a} = \frac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma\mathsf{a}}$. Note that the tensor multiplet fields are thus subject to two local gauge invariances,

$$B_{\mu\nu\mathsf{a}}(x) \to B_{\mu\nu\mathsf{a}}(x) + 2\,\partial_{[\mu}\Xi_{\nu]\mathsf{a}}(x)\,,\qquad P_{ij\mathsf{a}}(x) \to P_{ij\mathsf{a}}(x) + \mathrm{i}\xi_{ij\mathsf{a}}(x)\,. \tag{6.4}$$

Both these transformations appear in the supersymmetry commutation relation, which takes the form,

$$[\delta(\epsilon_1),\delta(\epsilon_2)] = 2(\bar\epsilon_2{}^i\gamma^\mu\epsilon_{1i} + \bar\epsilon_{2i}\gamma^\mu\epsilon_1{}^i)D_\mu + \delta(\Xi) + \delta(\xi)\,, \tag{6.5}$$

where the first term denotes the translation (covariantized with $A_\mu{}^M$ and $B_{\mu\nu\mathsf{a}}$ dependent terms) and the second and third one correspond to the transformations (6.4) with parameters,

$$\begin{aligned}
\Xi_{\mu\mathsf{a}} &= -\mathrm{i}\,(\bar\epsilon_2{}^i\gamma_\mu\epsilon_{1j} + \bar\epsilon_{2j}\gamma_\mu\epsilon_1{}^i)(P_{ik\mathsf{a}}\varepsilon^{kj} + \varepsilon_{ik}P^{kj}{}_\mathsf{a})\,,\\
\xi_{ij\mathsf{a}} &= 4\mathrm{i}\,(\bar\epsilon_2{}^k\gamma_\mu\epsilon_{1(i} + \bar\epsilon_{2(i}\gamma_\mu\epsilon_1{}^k)\,\varepsilon_{j)k}\,H^\mu{}_\mathsf{a} + 2\mathrm{i}\,(\bar\epsilon_2{}^k\gamma^\mu\epsilon_{1(i} + \bar\epsilon_{2(i}\gamma^\mu\epsilon_1{}^k)\;\partial_\mu P_{j)k\mathsf{a}}\\
&\quad - 2\mathrm{i}\,(\bar\epsilon_2{}^{(k}\gamma^\mu\epsilon_{1m} + \bar\epsilon_{2m}\gamma^\mu\epsilon_1{}^{(k})\,\varepsilon_{ik}\varepsilon_{jl}\,\partial_\mu P^{l)m}{}_\mathsf{a}\,.
\end{aligned} \tag{6.6}$$

Now we return to the vector multiplets with a deformation parametrized by the embedding tensor that couples the vector multiplets to a tensor multiplet background. The deformation is induced by changing the field strength tensors *and* the auxiliary fields in the supersymmetry transformation for Ω_i by

$$\begin{aligned}
F_{\mu\nu}{}^M &\longrightarrow \mathcal{H}_{\mu\nu}{}^M = \mathcal{F}_{\mu\nu}{}^M + gZ^{M,\mathsf{a}}B_{\mu\nu\mathsf{a}}\,,\\
Y_{ij}{}^M &\longrightarrow \mathcal{Y}_{ij}{}^M = Y_{ij}{}^M - \mathrm{i}gZ^{M,\mathsf{a}}P_{ij\mathsf{a}}\,.
\end{aligned} \tag{6.7}$$

[2]We use the notation of [25], with the exception of the tensor field which is rescaled by a factor 2. Note that the precise conventions are crucial for making contact with the tensor coupling to the vector multiplets, as employed in this paper (in particular, note (6.7)).

Observe that $\mathcal{Y}_{ij}{}^{M}$ is no longer pseudo-real. Since we insist on the fact that $\mathcal{H}_{\mu\nu}{}^{M}$ and $\mathcal{Y}_{ij}{}^{M}$ remain gauge invariant with respect to (6.4) we assume the following transformation rules for $A_\mu{}^M$ and $Y_{ij}{}^M$,

$$\delta A_\mu{}^M = -g Z^{M,\mathsf{a}} \Xi_{\mu\mathsf{a}} \,, \qquad \delta Y_{ij}{}^M = -g Z^{M,\mathsf{a}} \xi_{ij\mathsf{a}} \,. \tag{6.8}$$

Subsequently we evaluate the supersymmetry commutator on the vector multiplet fields acting on X^M, $A_\mu{}^M$ and $Y_{ij}{}^M$. For the moment, we assume non-trivial gaugings, generated by the same matrices $T_{MN}{}^P$ as before. We thus include order-g corrections to the supersymmetry variations of $\Omega_i{}^M$ and $Y_{ij}{}^M$. However, the closure of the supersymmetry commutator is non-trivial in view of the deformation (6.7) and the fact that the $T_{MN}{}^P$ do not satisfy the Jacobi identity. This will lead to new contributions to the supersymmetry commutator proportional to the tensor $Z^{M,\mathsf{a}}$. The result of an explicit calculation shows that these contributions can all be absorbed in the transformations parametrized by $\Xi_{\mu\mathsf{a}}$ and $\xi_{ij\mathsf{a}}$,

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = 2(\bar\epsilon_2{}^i\gamma^\mu\epsilon_{1i} + \bar\epsilon_{2i}\gamma^\mu\epsilon_1{}^i)D_\mu + \delta(\Lambda) + \delta(\Xi) + \delta(\xi) \,, \tag{6.9}$$

where the first term corresponds to a covariant translation (covariant with respect to vector and tensor gauge transformations). The corresponding parameters are equal to

$$\begin{aligned}
\Lambda^M =& 4\,(\bar X^M\,\bar\epsilon_2{}^i\epsilon_1{}^j\,\varepsilon_{ij} + X^M\,\bar\epsilon_{2i}\epsilon_{1j}\,\varepsilon^{ij})\,,\\
\Xi_{\mu\mathsf{a}} =& -2\,d_{\mathsf{a}NP}(A_\mu{}^N A_\nu{}^P + 2\,\eta_{\mu\nu}\bar X^N X^P)(\bar\epsilon_2{}^i\gamma^\nu\epsilon_{1i} + \bar\epsilon_{2i}\gamma^\nu\epsilon_1{}^i)\\
&- \mathrm{i}\,(\bar\epsilon_2{}^i\gamma_\mu\epsilon_{1j} + \bar\epsilon_{2j}\gamma_\mu\epsilon_1{}^i)(P_{ik\mathsf{a}}\varepsilon^{kj} + \varepsilon_{ik}P^{kj}{}_{\mathsf{a}})\,,\\
\xi_{ij\mathsf{a}} =& 4\,d_{\mathsf{a}NP}[X^N \overset{\leftrightarrow}{D}_\mu \bar X^P - \bar\Omega_l{}^N\gamma_\mu\Omega^{lP}\varepsilon_{k(i}(\bar\epsilon_2{}^k\gamma^\mu\epsilon_{1j)} + \bar\epsilon_{2j)}\gamma^\mu\epsilon_1{}^k)\\
&- 4\,d_{\mathsf{a}NP}\,\varepsilon^{kl}\,\bar\epsilon_{2k}\epsilon_{1l}\,(2\,X^N Y_{ij}{}^P - \bar\Omega_i{}^N\Omega_j{}^P)\\
&- 4\,d_{\mathsf{a}NP}\,\varepsilon_{kl}\,\bar\epsilon_2{}^k\epsilon_1{}^l\,(2\,\bar X^N Y^{ijP} - \varepsilon_{im}\varepsilon_{jn}\bar\Omega^{mN}\Omega^{nP})\\
&+ 4\mathrm{i}\,(\bar\epsilon_2{}^k\gamma_\mu\epsilon_{1(i} + \bar\epsilon_{2(i}\gamma_\mu\epsilon_1{}^k)\,\varepsilon_{j)k}\,H^\mu{}_{\mathsf{a}}\\
&+ 2\mathrm{i}\,(\bar\epsilon_2{}^k\gamma^\mu\epsilon_{1(i} + \bar\epsilon_{2(i}\gamma^\mu\epsilon_1{}^k)\;D_\mu P_{j)k\mathsf{a}}\\
&- 2\mathrm{i}\,(\bar\epsilon_2{}^{(k}\gamma^\mu\epsilon_{1m} + \bar\epsilon_{2m}\gamma^\mu\epsilon_1{}^{(k})\,\varepsilon_{ik}\varepsilon_{jl}\,D_\mu P^{l)m}{}_{\mathsf{a}}\,,
\end{aligned} \tag{6.10}$$

where use was made of the Bianchi identity (3.25). The important observation is that all the terms referring to the tensor multiplet fields in (6.10) are in precise agreement (up to the covariantizations) with (6.6). The remaining terms in Λ^M and $\Xi_{\mu\mathsf{a}}$ have already been found before in (4.14), while those in $\xi_{ij\mathsf{a}}$ are new.

What remains is to verify the closure on the fermion fields $\Omega_i{}^M$. In order to do so, we must first extend the tensor multiplets by incorporating non-abelian gauge couplings in (6.3). However, to keep matters simple, we will suppress non-abelian gauge interactions here and henceforth. In that case (6.3) is complete and the closure can be verified directly. As expected, the only possible terms that could affect the closure are the terms generated by the deformation (6.7). It is then a relatively straightforward calculation to verify that these terms cancel, so that we have indeed established the existence of an off-shell representation with both electric and magnetic charges present. Of course, these charges are then exclusively carried by hypermultiplets in the way that we have described before.

Let us now turn to the Lagrangian to see how the on-shell results of this paper can be obtained. The construction starts from the observation that, in the absence of the deformations (6.7), there exists a supersymmetric coupling between tensor and vector supermultiplets. For instance, such a coupling between the magnetic vector supermultiplets coupling and the tensor multiplets is described by the following Lagrangian,

$$\begin{aligned}\mathcal{L} \propto \Theta^{\Lambda\mathsf{a}}\Big\{ & G_{\mathsf{a}} X_\Lambda + \bar{G}_{\mathsf{a}} \bar{X}_\Lambda - \frac{1}{2}(P_{ij\mathsf{a}} Y^{ij}{}_\Lambda + P^{ij}{}_{\mathsf{a}} Y_{ij\Lambda}) \\ & + \bar{\Omega}^i{}_\Lambda \varphi_{i\mathsf{a}} + \bar{\Omega}_{i\Lambda} \varphi^i{}_{\mathsf{a}} - \frac{1}{2}\mathrm{i}\, \varepsilon^{\mu\nu\rho\sigma} B_{\mu\nu\mathsf{a}} F_{\rho\sigma\Lambda} \Big\}\,. \end{aligned} \tag{6.11}$$

In this Lagrangian the tensor multiplet fields act as Lagrange multipliers which would put the magnetic vector multiplet fields to zero. Instead, the on-shell theory that we are attempting to construct should lead to certain relations between the magnetic vector multiplet fields in terms of the other fields. Moreover, the Lagrangian (6.11) does not apply in the presence of the deformations. This suggests to make a number of modifications induced by the following shifts,

$$\begin{aligned} X_\Lambda &\longrightarrow X_\Lambda - F_\Lambda(X)\,, \\ \Omega_{i\Lambda} &\longrightarrow \Omega_{i\Lambda} - F_{\Lambda\Sigma}(X)\,\Omega_i{}^\Sigma\,, \\ F_{\mu\nu\Lambda} &\longrightarrow F_{\mu\nu\Lambda} - \frac{1}{4} g \Theta_\Lambda{}^{\mathsf{a}} B_{\mu\nu\mathsf{a}}\,, \\ Y_{ij\Lambda} &\longrightarrow Y_{ij\Lambda} + \frac{1}{4}\mathrm{i} g\, \Theta_\Lambda{}^{\mathsf{a}} P_{ij\mathsf{a}}\,, \end{aligned} \tag{6.12}$$

where $F(X)$ is the usual holomorphic function of the scalars X^Λ belonging to the *electric* vector multiplets. Note that the substitutions for $F_{\mu\nu\Lambda}$ and $Y_{ij\Lambda}$ coincide with the expressions for $\mathcal{H}_{\mu\nu\Lambda}$ and $\mathcal{Y}_{ij\Lambda}$ up to a factor of $\frac{1}{2}$. This is related to the fact that the fields $B_{\mu\nu\mathsf{a}}$ and $P_{ij\mathsf{a}}$ will appear quadratically in the Lagrangian upon performing the shifts (6.12). We note that the substitution for $Y_{ij\Lambda}$ is ambiguous in view of its pseudo-reality, while $Y_{ij\Lambda}$ and $Y^{ij}{}_\Lambda$ are assumed to acquire different shifts. Ultimately, the justification of these substitutions is, of course, given by the supersymmetry invariance of the resulting Lagrangian. Hence without further ado we now present the following extension of (6.11),

$$\begin{aligned}\mathcal{L} = -\frac{1}{4} g\, \Theta^{\Lambda\mathsf{a}}\Big\{ & G_{\mathsf{a}}[X_\Lambda - F_\Lambda(X)] + \bar{G}_{\mathsf{a}}[\bar{X}_\Lambda - \bar{F}_\Lambda(\bar{X})] \\ & - \frac{1}{2}\varepsilon^{ik}\varepsilon^{jl} P_{ij\mathsf{a}}\Big[Y_{kl\Lambda} + \frac{1}{4}\mathrm{i} g \Theta_\Lambda{}^{\mathsf{b}} P_{kl\mathsf{b}}\Big] - \frac{1}{2}\varepsilon_{ik}\varepsilon_{jl} P^{ij}{}_{\mathsf{a}}\Big[Y^{kl}{}_\Lambda - \frac{1}{4}\mathrm{i} g \Theta_\Lambda{}^{\mathsf{b}} P^{kl}{}_{\mathsf{b}}\Big] \\ & + \bar{\varphi}_{i\mathsf{a}}\Big[\Omega^i{}_\Lambda - \bar{F}_{\Lambda\Sigma}\Omega^{i\Sigma}\Big] + \bar{\varphi}^i_{\mathsf{a}}\Big[\Omega_{i\Lambda} - F_{\Lambda\Sigma}\Omega_i{}^\Sigma\Big] \\ & - \frac{1}{2}\mathrm{i}\,\varepsilon^{\mu\nu\rho\sigma} B_{\mu\nu\mathsf{a}}\Big[F_{\rho\sigma\Lambda} - \frac{1}{4} g \Theta_\Lambda{}^{\mathsf{b}} B_{\rho\sigma\mathsf{b}}\Big]\Big\}\,, \end{aligned} \tag{6.13}$$

which is invariant under the transformations (6.4), (6.8). This Lagrangian is the off-shell extension (in the abelian case) of (3.23), in view of the fact that the last term that contains the tensor gauge fields is identical. Clearly, the fields G_{a} and $\varphi^i{}_{\mathsf{a}}$ act as Lagrange multipliers, which determine the fields X_Λ and $\Omega_{i\Lambda}$ in the same way as before. The fields $P^{ij}{}_{\mathsf{a}}$ can be integrated out, just as the tensor fields $B_{\mu\nu\mathsf{a}}$.

The Lagrangian (6.13) must be combined with the Lagrangian (2.1) for the electric vector supermultiplets, in which we have to introduce the deformations (6.7). In the absence of magnetic charges (i.e. $\Theta^{\Lambda\mathsf{a}} = 0$), we thus obtain the standard result for electric charges.

In the presence of magnetic charges, the combined action leads to the field equation (3.27) for the tensor gauge field. For the field equation associated with $P_{ij}{}^{\mathsf{a}}$, we should first exhibit the deformation of the Lagrangian (2.5) that involves the auxiliary fields $Y_{ij}{}^{\Lambda}$. The correct way to introduce the deformation reads as follows,

$$\begin{aligned}\mathcal{L}_Y = &-\frac{1}{8}\mathrm{i}\varepsilon^{ik}\varepsilon^{jl}\Big[F_{\Lambda\Sigma}\,\mathcal{Y}_{ij}{}^{\Lambda}\mathcal{Y}_{kl}{}^{\Sigma} - F_{\Lambda\Gamma\Omega}\,\mathcal{Y}_{ij}{}^{\Lambda}\bar\Omega_k{}^{\Gamma}\Omega_l{}^{\Omega}\Big]\\ &+\frac{1}{8}\mathrm{i}\varepsilon_{ik}\varepsilon_{jl}\Big[\bar F_{\Lambda\Sigma}\,\mathcal{Y}^{ij\Lambda}\mathcal{Y}^{kl\Sigma} - \bar F_{\Lambda\Gamma\Omega}\,\mathcal{Y}^{ij\Lambda}\bar\Omega^{k\Gamma}\Omega^{l\Omega}\Big]\\ &-\frac{1}{32}N^{\Lambda\Sigma}\Big[\varepsilon^{ik}\varepsilon^{jl}F_{\Lambda\Gamma\Omega}\,\bar\Omega_i{}^{\Gamma}\Omega_j{}^{\Omega}\,F_{\Sigma\Xi\Delta}\,\bar\Omega_k{}^{\Xi}\Omega_l{}^{\Delta} + \varepsilon_{ik}\varepsilon_{jl}\bar F_{\Lambda\Gamma\Omega}\,\bar\Omega^{k\Gamma}\Omega^{l\Omega}\,\bar F_{\Sigma\Xi\Delta}\,\bar\Omega^{i\Xi}\Omega^{j\Delta}\Big]\\ &+\frac{1}{16}N^{\Lambda\Sigma}\,F_{\Lambda\Gamma\Omega}\,\bar\Omega_i{}^{\Gamma}\Omega_j{}^{\Omega}\,\bar F_{\Sigma\Xi\Delta}\,\bar\Omega^{i\Xi}\Omega^{j\Delta}\,. \end{aligned}\tag{6.14}$$

Here we observe that the structure of this expression is quite similar to the structure of (2.3), with the exception of the last term in the expression above which is separately consistent with respect to electric/magnetic duality. Actually this term cancels exactly against the last term of (2.4). The result of the field equations associated with the fields $P_{ij\Lambda}$ can now be determined and yields,

$$\Theta^{\Lambda\mathsf{a}}\bigg(\mathcal{Y}_{ij\Lambda} - \Big[F_{\Lambda\Sigma}\mathcal{Y}_{ij}{}^{\Sigma} - \frac{1}{2}F_{\Lambda\Sigma\Gamma}\bar\Omega_i{}^{\Sigma}\Omega_j{}^{\Gamma}\Big]\bigg) = 0\,. \tag{6.15}$$

This equation is in close analogy with the field equation (3.27) for the tensor gauge field.

To make contact with the on-shell results derived in this paper, we need the terms induced by the gauging for the hypermultiplet Lagrangian. Starting with the $2n$ independent vector supermultiplets, the terms of order g and g^2 will take the form,

$$\begin{aligned}\mathcal{L}_{g+g^2}\Big|_{\text{hypermultiplet}} = &+2g\,k_{AM}\,\big[\bar\gamma^{Ai}_{\alpha}\varepsilon_{ij}\,\bar\zeta^{\alpha}\Omega^{jM} + \gamma^{A}_{i\bar\alpha}\varepsilon^{ij}\,\bar\zeta^{\bar\alpha}\Omega_j{}^{M}\big]\\ &+2g\,\Big[\bar X^{M}t_{M}{}^{\gamma}{}_{\alpha}\,\bar\Omega_{\beta\gamma}\,\bar\zeta^{\alpha}\zeta^{\beta} + X^{M}t_{M}{}^{\bar\gamma}{}_{\bar\alpha}\,\Omega_{\bar\beta\bar\gamma}\,\bar\zeta^{\bar\alpha}\zeta^{\bar\beta}\Big]\\ &+g\,Y^{ijM}\mu_{ijM} - 2g^2k^{A}{}_{M}\,k^{B}{}_{N}\,g_{AB}\,X^{M}\bar X^{N}\,, \end{aligned}\tag{6.16}$$

where we include both electric and magnetic Killing potentials. In principle, one should modify this result by introducing the deformation (6.7). However, the effect of the deformation drops out in view of the fact that $Z^{M,\mathsf{a}}\mu_{ijM} = 0$, and the hypermultiplet Lagrangian is separately supersymmetric in the presence of the gauging.

Subsequently we note that the field $Y_{ij\Lambda}$ appears linearly in the combined Lagrangian, so that it acts as a Lagrange multiplier. Imposing, at the same time, the gauge condition that $P_{ij\mathsf{a}}$ is pesudo-real, we obtain the result,

$$\Theta^{\Lambda\mathsf{a}}\,P_{ij\mathsf{a}} = -4\,\mu_{ij}{}^{\Lambda}\,. \tag{6.17}$$

This introduces the correct supersymmetry variation of the fermion field $\Omega_i{}^{\Lambda}$, because $\mathcal{Y}_{ij}{}^{\Lambda} = Y_{ij}{}^{\Lambda} + 2g\mathrm{i}\,\mu_{ij}{}^{\Lambda}$. Substituting this last expression into (6.14) leads then to additional terms in (6.16) linear and quadratic in the magnetic Killing potentials ${\mu_{ij}}^{\Lambda}$. These terms coincide with the corresponding terms given in (5.28) and (5.30).

It should be interesting to further explore the properties and possible applications of this off-shell formulation. An obvious question concerns the existence of a non-abelian version.

7. Summary and discussion

In this paper we presented Lagrangians and supersymmetry transformations for a general supersymmetric system of vector multiplets and hypermultiplets in the presence of both electric and magnetic charges. The results were verified to all orders and are consistent with results known in the literature that are based on purely electric charges. The closure of the supersymmetry algebra, is realized on shell, but in the previous section we have indicated how an off-shell representation can be defined consisting of vector and tensor supermultiplets.

Before discussing possible implications of these results, let us first summarize the terms induced by the gauging. We first present the combined supersymmetry variations. First of all, we have the original transformations in the absence of the gauging, where space-time derivatives are replaced by gauge-covariant derivatives and where the abelian field strengths ${F_{\mu\nu}}^{\Lambda}$ are replaced by the covariant field strengths ${\mathcal{H}_{\mu\nu}}^{\Lambda}$. We will not repeat the corresponding expressions here, but we present the other terms in the transformation rules that are induced by the gauging. They read as follows,

$$
\begin{aligned}
\delta_g {\Omega_i}^{\Lambda} &= -2g\, {T_{NP}}^{\Lambda}\, \bar X^N X^P\, \varepsilon_{ij}\, \epsilon^j + 2\,\mathrm{i}g\, {\mu_{ij}}^{\Lambda}\epsilon^j\,,\\
\delta_g \zeta^{\alpha} &= 2\,gX^M\, {k^A}_M V^{\alpha}_{Ai}\, \varepsilon^{ij}\epsilon_j\,,\\
\delta_g {Y_{ij}}^{\Lambda} &= -4g{T_{MN}}^{\Lambda}\Big[\bar\Omega_{(i}{}^M\epsilon^k\varepsilon_{j)k}\bar X^N - \bar\Omega^{kM}\epsilon_{(i}\varepsilon_{j)k}X^N\Big] + 4\mathrm{i}gk^{A\Lambda}\Big[\varepsilon_{k(i}\gamma_{j)\bar\alpha A}\bar\epsilon^k\zeta^{\bar\alpha} + \varepsilon_{k(i}\bar\epsilon_{j)}\zeta^{\alpha}\bar\gamma^k_{\alpha A}\Big],\\
\delta B_{\mu\nu\mathsf{a}} &= -2t_{\mathsf{a}M}{}^P\Omega_{PN}\left(A_{[\mu}{}^M\,\delta A_{\nu]}{}^N - \bar X^M\bar\Omega_i{}^N\gamma_{\mu\nu}\epsilon^i - X^M\bar\Omega^{iN}\gamma_{\mu\nu}\epsilon_i\right)\\
&\quad - 4\mathrm{i}k^A{}_{\mathsf{a}}\,\left[\gamma_{Ai\bar\alpha}\,\bar\zeta^{\bar\alpha}\gamma_{\mu\nu}\epsilon^i - \bar\gamma^i_{A\alpha}\,\bar\zeta^{\alpha}\gamma_{\mu\nu}\epsilon_i\right]\,.
\end{aligned}
\tag{7.1}
$$

Likewise we will not repeat the original Lagrangians (2.1) and (5.2) for the vector multiplets and hypermultiplets, respectively, modified by the replacement of space-time derivatives by gauge-covariant ones, and field strengths by the covariant field strengths ${\mathcal{H}_{\mu\nu}}^{\Lambda}$. The Lagrangian (3.23) remains unchanged. The additional terms induced by the gauging that are linear in g take the following form,

$$
\begin{aligned}
\mathcal{L}_g = &-\frac{1}{2}\mathrm{i}g\,\Omega_{MQ}T_{PN}{}^Q\,\left[\varepsilon^{ij}\,\bar\Omega_i{}^M\Omega_j{}^P\bar X^N - \varepsilon_{ij}\,\bar\Omega^{iM}\Omega^{jP}X^N\right]\\
&-\frac{1}{4}g\,\left[F_{\Lambda\Sigma\Gamma}\,\mu^{ij\Lambda}\,\bar\Omega_i{}^{\Sigma}\Omega_j{}^{\Gamma} + \bar F_{\Lambda\Sigma\Gamma}\,\mu_{ij}{}^{\Lambda}\,\bar\Omega^{i\Sigma}\Omega^{j\Gamma}\right]\,,\\
&+2g\,k_{AM}\,\left[\bar\gamma^{Ai}_{\alpha}\varepsilon_{ij}\,\bar\zeta^{\alpha}\Omega^{jM} + \gamma^{A}_{i\bar\alpha}\varepsilon^{ij}\,\bar\zeta^{\bar\alpha}\Omega_j{}^M\right]\\
&+2g\left[\bar X^M t_M{}^{\gamma}{}_{\alpha}\,\bar\Omega_{\beta\gamma}\,\bar\zeta^{\alpha}\zeta^{\beta} + X^M t_M{}^{\bar\gamma}{}_{\bar\alpha}\,\Omega_{\bar\beta\bar\gamma}\,\bar\zeta^{\bar\alpha}\zeta^{\bar\beta}\right]\\
&+g\,Y^{ij\Lambda}\left[\mu_{ij\Lambda} + \frac{1}{2}(F_{\Lambda\Sigma} + \bar F_{\Lambda\Sigma})\,\mu_{ij}{}^{\Sigma}\right]\,.
\end{aligned}
\tag{7.2}
$$

The terms of order g^2 correspond to a scalar potential proportional to g^2 and are given by

$$\begin{aligned}\mathcal{L}_{g^2} &= \mathrm{i}g^2\,\Omega_{MN}\,T_{PQ}{}^M X^P \bar{X}^Q\,T_{RS}{}^N \bar{X}^R X^S \\ &\quad -2g^2 k^A{}_M\,k^B{}_N\,g_{AB}\,X^M\bar{X}^N - \frac{1}{2}g^2\,N_{\Lambda\Sigma}\,\mu_{ij}{}^\Lambda\,\mu^{ij\Sigma}\,. \end{aligned} \tag{7.3}$$

Eliminating the auxiliary fields $Y_{ij}{}^\Lambda$ gives rise to the following expressions. The terms linear in g read,

$$\begin{aligned}\mathcal{L}_g &= -\frac{1}{2}\mathrm{i}g\,\Omega_{MQ}T_{PN}{}^Q\,\left[\varepsilon^{ij}\,\bar{\Omega}_i{}^M\Omega_j{}^P\bar{X}^N - \varepsilon_{ij}\,\bar{\Omega}^{iM}\Omega^{jP}X^N\right] \\ &\quad + 2g\,k_{AM}\,\left[\bar{\gamma}^{Ai}_\alpha\varepsilon_{ij}\,\bar{\zeta}^\alpha\Omega^{jM} + \gamma^A_{i\bar{\alpha}}\varepsilon^{ij}\,\bar{\zeta}^{\bar{\alpha}}\Omega_j{}^M\right] \\ &\quad + 2g\left[\bar{X}^M t_M{}^\gamma{}_\alpha\,\bar{\Omega}_{\beta\gamma}\,\bar{\zeta}^\alpha\zeta^\beta + X^M t_M{}^{\bar{\gamma}}{}_{\bar{\alpha}}\,\Omega_{\bar{\beta}\bar{\gamma}}\,\bar{\zeta}^{\bar{\alpha}}\zeta^{\bar{\beta}}\right] \\ &\quad - \frac{1}{2}\mathrm{i}gN^{\Lambda\Sigma}\,F_{\Sigma\Gamma\Xi}\,\bar{\Omega}_i{}^\Gamma\Omega_j{}^\Xi\,\left[\mu^{ij}{}_\Lambda + \bar{F}_{\Lambda\Delta}\,\mu^{ij\Delta}\right] \\ &\quad + \frac{1}{2}\mathrm{i}gN^{\Lambda\Sigma}\,\bar{F}_{\Sigma\Gamma\Xi}\,\Omega^{i\Gamma}\Omega^{j\Xi}\,\left[\mu_{ij\Lambda} + F_{\Lambda\Delta}\,\mu_{ij}{}^\Delta\right]\,. \end{aligned} \tag{7.4}$$

The resulting potential, which is proportional to g^2, follows from

$$\begin{aligned}\mathcal{L}_{g^2} &= \mathrm{i}g^2\,\Omega_{MN}\,T_{PQ}{}^M X^P\bar{X}^Q\,T_{RS}{}^N\bar{X}^R X^S - 2g^2k^A{}_M\,k^B{}_N\,g_{AB}\,X^M\bar{X}^N \\ &\quad - 2\,g^2\left[\mu^{ij}{}_\Lambda + F_{\Lambda\Gamma}\,\mu^{ij\Gamma}\right]\,N^{\Lambda\Sigma}\left[\mu_{ij\Sigma} + \bar{F}_{\Sigma\Xi}\,\mu_{ij}{}^\Xi\right]\,. \end{aligned} \tag{7.5}$$

Provided the embedding tensor is treated as a spurionic quantity, both these expressions are invariant under electric/magnetic duality transformations.

The same phenomenon can be seen in the supersymmetry variation of the vector multiplet fermions, upon integrating out the fields $Y_{ij}{}^\Lambda$. Up to terms quadratic in the fermions, this variation reads,

$$\begin{aligned}\delta\Omega_i{}^\Lambda &= 2\not{D}X^\Lambda\epsilon_i + \frac{1}{2}\gamma^{\mu\nu}\mathcal{H}^{-}_{\mu\nu}{}^\Lambda\varepsilon_{ij}\epsilon^j \\ &\quad - 2g\,T_{NP}{}^\Lambda\,\bar{X}^N X^P\,\varepsilon_{ij}\,\epsilon^j - 4\,gN^{\Lambda\Sigma}(\mu_{ij\Sigma} + \bar{F}_{\Sigma\Gamma}\,\mu_{ij}{}^\Gamma)\epsilon^j\,, \end{aligned} \tag{7.6}$$

where the term of order g is consistent with electric/magnetic duality.

The above results have many applications. A relatively simple one concerns the Fayet-Iliopoulos terms, which are the integration constants of the Killing potentials $\mu^{ij}{}_M$. This enables us to truncate the above expressions by setting the embedding tensor to zero, while still retaining the constants $g\mu^{ij}{}_M$. In that case all effects of the gauging are suppressed and one is left with a potential accompanied by fermionic masslike terms,

$$\begin{aligned}\mathcal{L}_{\mathrm{FI}} &= -\frac{1}{2}\mathrm{i}gN^{\Lambda\Sigma}\,F_{\Sigma\Gamma\Xi}\,\bar{\Omega}_i{}^\Gamma\Omega_j{}^\Xi\,\left[\mu^{ij}{}_\Lambda + \bar{F}_{\Lambda\Delta}\,\mu^{ij\Delta}\right] \\ &\quad + \frac{1}{2}\mathrm{i}gN^{\Lambda\Sigma}\,\bar{F}_{\Sigma\Gamma\Xi}\,\Omega^{i\Gamma}\Omega^{j\Xi}\,\left[\mu_{ij\Lambda} + F_{\Lambda\Delta}\,\mu_{ij}{}^\Delta\right] \\ &\quad - 2\,g^2\left[\mu^{ij}{}_\Lambda + F_{\Lambda\Gamma}\,\mu^{ij\Gamma}\right]\,N^{\Lambda\Sigma}\left[\mu_{ij\Sigma} + \bar{F}_{\Sigma\Xi}\,\mu_{ij}{}^\Xi\right]\,. \end{aligned} \tag{7.7}$$

The above expression transforms as a function under electric/magnetic duality provided that the $\mu^{ij}{}_M$ are treated as spurionic quantities transforming as a $2n$-vector under

$\mathrm{Sp}(2n,\mathbb{R})$. To show this one makes use of the transformation rules (2.24) for the second and third derivatives of the holomorphic function $F(X)$. The last term in (7.7) corresponds to minus the potential, which is positive definite (assuming positive $N_{\Lambda\Sigma}$). The Lagrangian is a generalization of the Lagrangian presented in [26], where it was also shown how the potential can lead to spontaneous partial supersymmetry breaking when ${\mu_{ij}}^{\Lambda} \neq 0$. Note that the hypermultiplets play only an ancillary role here, as they decouple from the vector multiplets.

Most of the possible applications can be found in the context of supergravity, where they will be useful for constructing low-energy effective actions associated with string compactifications in the presence of fluxes (see, e.g. [27]). In principle it is straightforward to extend our results to the case of local supersymmetry. The target space of the vector multiplets should then be restriced to a special Kähler cone (this requires that $F(X)$ be a homogeneous function of second degree), and the hypermultiplet scalars should coordinatize a hyperkähler cone. Furthermore the various formulae for the action and the supersymmetry transformation rules should be evaluated in the presence of a superconformal background, so that the action and transformation rules will also involve the superconformal fields. This has not yet been worked out in detail for $N = 2$ supergravity, although it is in principle straightforward. In view of the fact that gaugings of $N = 4$ and $N = 8$ supergravity have already been worked out using the same formalism as in this paper [2, 3], no complications are expected. Note that Fayet-Iliopoulos terms do not exist in $N = 2$ supergravity because the Killing potentials cannot contain arbitrary integration constants as those would break the scale invariance of the hyperkähler cone.

The potential is rather independent of all these details, although it must be rewritten in terms of the appropriate quantities, as was for instance demonstrated in [24]. It was already shown in [1] that the theory simplifies considerably for abelian gaugings where ${T_{MN}}^{P} = 0$ and where the potential is exclusively generated by the hypermultiplet charges. Making use of the steps described in [24], it is rather straightforward to derive the potential (as was already foreseen in [1]), which takes precisely the form conjectured quite some time ago (c.f. eq. (3.16) in [28]).

Another application concerns domain wall solutions. In [29] such solutions were studied in $N = 2$ supergravity with both electric and magnetic charges. The transformation rules postulated in that work are in qualitative agreement with the ones established in this paper, at least as far as the terms are concerned that are relevant for the potential (observe that a magnetic gauge field was absent). A more precise comparison again requires the extention of our results to the case of local supersymmetry.

Acknowledgments

We thank Stefan Vandoren for valuable discussions. The work of M.d.V. is part of the research programme of the 'Stichting voor Fundamenteel Onderzoek der Materie (FOM)', which is financially supported by the 'Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO)'.

This work is also partly supported by NWO grant 047017015, EU contract MRTN-CT-2004-005104 and INTAS contract 03-51-6346.

References

[1] B. de Wit, H. Samtleben and M. Trigiante, *Magnetic charges in local field theory*, *JHEP* **09** (2005) 016 [hep-th/0507289].

[2] J. Schon and M. Weidner, *Gauged $N = 4$ supergravities*, *JHEP* **05** (2006) 034 [hep-th/0602024].

[3] B. de Wit, H. Samtleben and M. Trigiante, *The maximal $D = 4$ supergravities*, *JHEP* **06** (2007) 049 [arXiv:0705.2101].

[4] J. Louis and A. Micu, *Type II theories compactified on Calabi-Yau threefolds in the presence of background fluxes*, *Nucl. Phys.* **B 635** (2002) 395 [hep-th/0202168].

[5] G. Dall'Agata, R. D'Auria, L. Sommovigo and S. Vaula, *$D = 4$, $N = 2$ gauged supergravity in the presence of tensor multiplets*, *Nucl. Phys.* **B 682** (2004) 243 [hep-th/0312210].

[6] L. Sommovigo and S. Vaula, *$D = 4$, $N = 2$ supergravity with abelian electric and magnetic charge*, *Phys. Lett.* **B 602** (2004) 130 [hep-th/0407205].

[7] R. D'Auria, L. Sommovigo and S. Vaula, *$N = 2$ supergravity lagrangian coupled to tensor multiplets with electric and magnetic fluxes*, *JHEP* **11** (2004) 028 [hep-th/0409097].

[8] B. de Wit, P.G. Lauwers and A. Van Proeyen, *Lagrangians of $N = 2$ supergravity-matter systems*, *Nucl. Phys.* **B 255** (1985) 569.

[9] B. de Wit, J.W. van Holten and A. Van Proeyen, *Structure of $N = 2$ supergravity*, *Nucl. Phys.* **B 184** (1981) 77 [*Erratum ibid.* **B222** (1983) 516].

[10] B. de Wit and A. Van Proeyen, *Potentials and symmetries of general gauged $N = 2$ supergravity: Yang-Mills models*, *Nucl. Phys.* **B 245** (1984) 89.

[11] B. de Wit, *Electric-magnetic duality in supergravity*, *Nucl. Phys.* **101** *(Proc. Suppl.)* (2001) 154 [hep-th/0103086].

[12] B. de Wit, *$N = 2$ electric-magnetic duality in a chiral background*, *Nucl. Phys.* **49** *(Proc. Suppl.)* (1996) 191 [hep-th/9602060].

[13] M.K. Gaillard and B. Zumino, *Duality rotations for interacting fields*, *Nucl. Phys.* **B 193** (1981) 221.

[14] S. Cecotti, S. Ferrara and L. Girardello, *Geometry of type-II superstrings and the moduli of superconformal field theories*, *Int. J. Mod. Phys.* **A 4** (1989) 2475.

[15] B. de Wit, H. Samtleben and M. Trigiante, *The maximal $D = 5$ supergravities*, *Nucl. Phys.* **B 716** (2005) 215 [hep-th/0412173].

[16] B. de Wit and H. Samtleben, *Gauged maximal supergravities and hierarchies of nonabelian vector-tensor systems*, *Fortschr. Phys.* **53** (2005) 442 [hep-th/0501243].

[17] J. De Jaegher, B. de Wit, B. Kleijn and S. Vandoren, *Special geometry in hypermultiplets*, *Nucl. Phys.* **B 514** (1998) 553 [hep-th/9707262].

[18] B. de Wit, B. Kleijn and S. Vandoren, *Superconformal hypermultiplets*, *Nucl. Phys.* **B 568** (2000) 475 [hep-th/9909228].

[19] G. Sierra and P.K. Townsend, *The gauge invariant $N = 2$ supersymmetric sigma model with general scalar potential*, *Nucl. Phys.* **B 233** (1984) 289.

[20] C.M. Hull, A. Karlhede, U. Lindström and M. Roček, *Nonlinear sigma models and their gauging in and out of superspace*, *Nucl. Phys.* **B 266** (1986) 1.

[21] J.A. Bagger, A.S. Galperin, E.A. Ivanov and V.I. Ogievetsky, *Gauging $N = 2$ sigma models in harmonic superspace*, *Nucl. Phys.* **B 303** (1988) 522.

[22] R. D'Auria, S. Ferrara and P. Fré, *Special and quaternionic isometries: general couplings in $N = 2$ supergravity and the scalar potential*, *Nucl. Phys.* **B 359** (1991) 705.

[23] L. Andrianopoli et al., *$N = 2$ supergravity and $N = 2$ super Yang-Mills theory on general scalar manifolds: symplectic covariance, gaugings and the momentum map*, *J. Geom. Phys.* **23** (1997) 111 [`hep-th/9605032`].

[24] B. de Wit, M. Roček and S. Vandoren, *Gauging isometries on hyperKähler cones and quaternion-Kähler manifolds*, *Phys. Lett.* **B 511** (2001) 302 [`hep-th/0104215`].

[25] B. de Wit and F. Saueressig, *Off-shell $N = 2$ tensor supermultiplets*, *JHEP* **09** (2006) 062 [`hep-th/0606148`].

[26] I. Antoniadis, H. Partouche and T.R. Taylor, *Spontaneous breaking of $N = 2$ global supersymmetry*, *Phys. Lett.* **B 372** (1996) 83 [`hep-th/9512006`].

[27] M. Graña, *Flux compactifications in string theory: a comprehensive review*, *Phys. Rept.* **423** (2006) 91 [`hep-th/0509003`].

[28] J. Michelson, *Compactifications of type IIB strings to four dimensions with non-trivial classical potential*, *Nucl. Phys.* **B 495** (1997) 127 [`hep-th/9610151`].

[29] K. Behrndt, G. Lopes Cardoso and D. Lüst, *Curved BPS domain wall solutions in four-dimensional $N = 2$ supergravity*, *Nucl. Phys.* **B 607** (2001) 391 [`hep-th/0102128`].

JHEP08(2007)064